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DEFORMATIONS OF COMPACT COMPLEX MANIFOLDS

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ABSTRACT

We look at some elementary aspects of the deformation theory of compact complex manifolds. This is done in a framework that examines the local structure of the space of almost complex structures on a given smooth manifold. We will define for this purpose Kodaira-Spencer classes and the notion of their integrability. We prove the theorem of existence, which guarantees that arbitrary Kodaira-Spencer classes are integrable under some vanishing condition. We also prove the Tian-Todorov theorem, which similarly guarantees the integrability of arbitrary Kodaira-Spencer classes, this time in the setting of Calabi-Yau manifolds. A brief exposition of an alternative framework for this deformation theory, mainly due to Kodaira and Spencer, is given. In the Appendix, the Newlander-Nirenberg theorem is proven.

Keywords: Complex manifolds, deformations, Kodaira-Spencer classes, theorem of existence, Calabi-Yau manifolds, Tian-Todorov theorem, Newlander-Nirenberg theorem.

DÉFORMATIONS DES VARIÉTÉS COMPLEXES COMPACTES

RÉSUMÉ

Nous examinons quelques aspects élémentaires de la théorie de la déformation des variétés complexes compactes. Cela est fait dans un cadre où l'on s'intéresse à la structure locale de l'espace des structures presque complexes sur une variété lisse donnée. Nous définissons dans ce but les classes de Kodaira-Spencer et la notion de leur intégrabilité. Nous démontrons le théorème d'existence, qui garantit l'intégrabilité d'une classe de Kodaia-Spencer arbitraire pourvue qu'une certaine condition d'annulation soit satisfaite. Nous démontrons également le théorème de Tian-Todorov qui, de même façon, garantit l'intégrabilité d'une classe de Kodaira-Spencer arbitraire, cette fois-ci dans le contexte des variétés de Calabi-Yau. Une brève présentation d'une différente approche à cette théorie de la déformation, due principalement à Kodaira et Spencer, est donnée. Dans l'appendice, nous donnons une démonstration du théorème de Newlander-Nirenberg.

Mots clés: Variétés complexes, déformations, classes de Kodaira-Spencer, théorème d'existence, variétés de Calabi-Yau, théorème de Tian-Todorov, théorème de Newlander-Nirenberg.

INTRODUCTION

A complex manifold can be defined as a smooth manifold together with a complex structure. Some complex structures, such as the standard one on complex projective space, appear to be isolated and rigid, while others, such as those on the torus, seem to exist among and close to others. It is then natural to try to think in geometric terms about the collection of all complex structures on a given smooth manifold. This idea originates with Riemann in his work on Riemann surfaces in the 19^{th} century and leads to the theory of moduli spaces. Deformation theory is a part of this larger theory that is concerned with local questions. In what follows we will flesh out a framework for the deformation theory of compact complex manifolds. The main thrust of this theory came in the middle of the 20^{th} century with the works of, among others, Kodaira, Kuranishi, Nirenberg and Spencer.

Given a complex manifold (M, J), a one parameter deformation of J is a curve in the space of almost complex structures on M starting at J and passing only through integrable almost complex structures. The "first order variation" of such a curve is a Kodaira-Spencer class, i.e. an element of the sheaf of germs of holomorphic vector bundles on (M, J). A Kodaira-Spencer class is said to be integrable if it is associated in this way to a one parameter deformation of J. Giving a description of all such classes is the focal point of deformation theory. We will prove two results on integrability: the theorem of existence and the Tian-Todorov theorem.

In Chapter 1, we recall some elementary definitions and results in complex geometry that we will need in the course of our endeavor.

In Chapter 2, we briefly survey the analytic tools that we will use. Chief among them is the theory of elliptic linear differential operators.

In Chapter 3, we describe in geometric terms the space of almost complex structures on a given smooth manifold. As we will see, this is a Fréchet manifold.

In Chapter 4, we derive the so-called Maurer-Cartan condition. We define one parameter deformations, infinitesimal deformations, Kodaira-Spencer classes and integrability.

In Chapter 5, we prove the theorem of existence. We do so in two different ways: first, with a power series method, as done by Kodaira, Nirenberg and Spencer (Kodaira *et al.*, 1958), and then by a method using the inverse function theorem in Banach spaces due to Kuranishi (Kuranishi, 1965).

In Chapter 6, we give a proof of the Tian-Todorov theorem. The proof is essentially a simple application of the Tian-Todorov lemma (also proven) and the techniques that are introduced in Chapter 5.

In Chapter 7, an alternative framework for the same deformation theory is presented. This framework is in fact the one that was used by the two pioneers of the subject, Kodaira and Spencer.

In Appendix A, a proof of the Newlander-Nirenberg theorem, due to Malgrange (Malgrange, 1969), is given.

CHAPTER I

BASIC COMPLEX GEOMETRY

We give a basic introduction to complex geometry, based mostly on the exposition in (Kobayashi et Nomizu, 1969). The main purpose is to fix the notation for the rest of the document.

1.1 Almost complex structures

1.1.1 On a vector space

An *almost complex structure* (ACS) on a real vector space V is an endomorphism $J : V \to V$ such that $J^2 = -\mathrm{Id}$. Importantly, V can be given the structure of a complex vector space by defining $\sqrt{-1}v = J(v)$. The pair (V, J) designates this complex vector space. If n is the (complex) dimension of this new vector space, then the (real) dimension of V is $2n$. Thus a real vector space that admits an ACS is necessarily of even dimension.

Example 1.1.1. Take $V = \mathbf{R}^{2n}$ and identify it with $\mathbf{C}^n$ via

$$(x^1, \ldots, x^n, y^1, \ldots, y^n) \mapsto (x^1 + \sqrt{-1}y^1, \ldots, x^n + \sqrt{-1}y^n).$$

Put $J(X) = \sqrt{-1}X$. Then J is an ACS on $\mathbf{R}^{2n}$, called the standard ACS. It has the matrix

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{1.1.1}$$

For the remainder of this subsection, J is an ACS on a real vector space V of dimension $2n$.

Proposition 1.1.1. *Let V' be real vector space and J' an ACS on V'. An* **R***-linear map $\omega : V \to V'$ is* **C***-linear as a map from (V, J) to (V', J') if and only if $\omega J = J'\omega$.*

Proof. This is obvious from the fact that applying J and J' correspond to multiplying by $\sqrt{-1}$ in the spaces (V, J) and (V', J') respectively. ■

Proposition 1.1.2. *There is a basis of V of the form $\{X_1, \ldots, X_n, J(X_1), \ldots, J(X_n)\}$.*

Proof. The vector space (V, J) has complex dimension n. Let B be a basis for it. Then it is obvious that $B \cup J(B)$ is a basis for V seen as a real vector space. ■

Let $V^{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of V. Conjugation on $V^{\mathbf{C}}$ is defined by $\overline{X \otimes z} = X \otimes \overline{z}$. Clearly this is a real automorphism of $V^{\mathbf{C}}$. We extend J by linearity to an endomorphism of $V^{\mathbf{C}}$, also denoted by J. Since $J^2 = -\operatorname{Id}$ and the characteristic polynomial of J is real, the eigenvalues of J are $\sqrt{-1}$ and $-\sqrt{-1}$ and their algebraic multiplicities are both n. Let $V^{1,0}$ and $V^{0,1}$ be, respectively, the eigenspaces of $\sqrt{-1}$ and $-\sqrt{-1}$.

Proposition 1.1.3. *We have:*

- **(i)** $V^{1,0} = \{X - \sqrt{-1}J(X) \ : \ X \in V\}$ *and* $V^{0,1} = \{X + \sqrt{-1}J(X) \ : \ X \in V\}$;
- **(ii)** $V^{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$;
- **(iii)** $Z \mapsto \overline{Z}$ *is a real self-inverse isomorphism between* $V^{1,0}$ *and* $V^{0,1}$.

Proof. It is clear that

$$V^{\mathbf{C}} = \{X - \sqrt{-1}J(X) \ : \ X \in V\} \oplus \{X + \sqrt{-1}J(X) \ : \ X \in V\}$$

and that the subspaces of $V^{\mathbf{C}}$ appearing on the RHS are respectively subspaces of $V^{1,0}$ and $V^{0,1}$. From this, (ii), (i) and (iii) follow, in this order. ∎

One consequence of the above proposition is that there are **C**-isomorphisms from (V, J) to $V^{1,0}$ and from $(V, -J)$ to $V^{0,1}$ which, respectively, send $X \in V$ to $X - \sqrt{-1}J(X) \in V^{1,0}$ and $X + \sqrt{-1}J(X) \in V^{0,1}$.

There is a natural ACS that can be defined on V^*, which we also denote by J. Given $\varphi \in V^*$, $J(\varphi)$ is the element of V^* such that

$$J(\varphi)(X) = \varphi(J(X))$$

for all $X \in V$.

Consider now the complexification of the dual space, $(V^*)^{\mathbf{C}}$. It is an easy matter to establish that this space is isomorphic to $(V^{\mathbf{C}})^*$, and we use $V^{*\mathbf{C}}$ to denote either of those spaces. Let $V_{1,0}$ and $V_{0,1}$ be, respectively, the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of $J : V^* \to V^*$. We have the decomposition

$$V^{*\mathbf{C}} = V_{1,0} \oplus V_{0,1}.$$

The proof of the following proposition is trivial.

Proposition 1.1.4. *We have*

$$V_{1,0} = \{\varphi \in V^{*\mathbf{C}} \ : \ \varphi(Z) = 0 \ \ \forall \ Z \in V^{0,1}\}$$
$$V_{0,1} = \{\varphi \in V^{*\mathbf{C}} \ : \ \varphi(Z) = 0 \ \ \forall \ Z \in V^{1,0}\}.$$

∎

We now examine the exterior algebra $\bigwedge V^{*\mathbf{C}}$. The exterior algebras $\bigwedge V_{1,0}$ and $\bigwedge V_{0,1}$ may naturally be considered as subalgebras of $\bigwedge V^{*\mathbf{C}}$. We define

$$\bigwedge^{p,q} V^{*\mathbf{C}} = \bigwedge^{p} V_{1,0} \wedge \bigwedge^{q} V_{0,1}$$

for non-negative integers p and q. By convention, $\bigwedge^{p,q} V^{*\mathbf{C}} = 0$ for $p < 0$ or $q < 0$. Note that $\bigwedge^{p,q} V^{*\mathbf{C}} = 0$ if $p > n$ or $q > n$. The proof of the next proposition is straightforward.

Proposition 1.1.5. *Given a non-negative integer r, the r^{th} exterior product, $\bigwedge^r V^{*\mathbf{C}}$, decomposes as*

$$\bigwedge^{r} V^{*\mathbf{C}} = \bigoplus_{p+q=r} \bigwedge^{p,q} V^{*\mathbf{C}},$$

so that

$$\bigwedge V^{*\mathbf{C}} = \bigoplus_{r=0}^{2n} \bigwedge^{r} V^{*\mathbf{C}} = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \bigwedge^{p,q} V^{*\mathbf{C}}.$$

Furthermore, the natural extension of conjugation on $V^{*\mathbf{C}}$ to $\bigwedge V^{*\mathbf{C}}$ is a real automorphism of $\bigwedge V^{*\mathbf{C}}$ and sends $\bigwedge^{p,q} V^{*\mathbf{C}}$ to $\bigwedge^{q,p} V^{*\mathbf{C}}$. ■

Let $\{e^1, \ldots, e^n\}$ be a basis for $V_{1,0}$. Then $\{\overline{e}^1, \ldots, \overline{e}^n\}$, $\overline{e}^i = \overline{e^i}$, is a basis for $V_{0,1}$ (item (iii) of Proposition 1.1.3). A basis for $\bigwedge^{p,q} V^{*\mathbf{C}}$ is

$$\{e^{j_1} \wedge \cdots \wedge e^{j_p} \wedge \overline{e}^{k_1} \wedge \cdots \wedge \overline{e}^{k_q} \ : 1 \leq j_1 < j_2 < \ldots < j_p \leq n \ ,$$
$$1 \leq k_1 < k_2 < \ldots < k_q \leq n\}.$$

From this, we readily establish that there is a natural isomorphism

$$\bigwedge^{p,q} V^{*\mathbf{C}} \simeq \bigwedge^{p,0} V^{*\mathbf{C}} \otimes_{\mathbf{C}} \bigwedge^{0,q} V^{*\mathbf{C}}.$$

1.1.2 On a manifold

We will adapt the definitions and results of the previous subsection to the tangent bundle of a given manifold, proceeding fiberwise. Let M be a C^∞ manifold. An *almost* *complex structure* (ACS) on M is a smooth section J of $TM \otimes T^*M$ on M such that $J^2 = -\operatorname{Id}$. Thus for every $x \in M$, J_x is an ACS on the vector space T_xM. The pair (M, J) is called an *almost complex manifold* (ACM). If V is a real vector space of even dimension then, when seeing V as a C^∞ manifold, an ACS on V as defined in the previous section gives an ACS in the sense just defined.

Proposition 1.1.6. *Let (M, J) be an ACM. The dimension of M is even and M is orientable.*

Proof. That the dimension of M is even follows from the fact that, for any $x \in M$, T_xM is of even dimension since it admits the ACS (as defined in the previous section) J_x. As for M being orientable, it is an easy matter to show that assigning to each $x \in M$ a basis of the form $X_1, \dots, X_n, J_x(X_1), \dots, J_x(X_n)$ where $n = \frac{1}{2}\dim M$ (see Proposition 1.1.2), determines a unique orientation on M. ■

The orientation described above is called the *natural orientation* on M. If (M', J') is another ACM, a C^∞ function f of an open subset of M into M' is said to be *almost complex* if, for each x in its domain, $Df|_x \circ J = J' \circ Df|_x$. The next proposition will prove useful shortly.

Proposition 1.1.7. *Let $f : U \subseteq \mathbf{C}^n \to \mathbf{C}^m$ be a C^∞ function, U open in $\mathbf{C}^n$. Identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ and $\mathbf{C}^m$ with $\mathbf{R}^{2m}$ as in Example 1.1.1. Let J and J' be, respectively, the standard ACS's on $\mathbf{R}^{2n}$ and $\mathbf{R}^{2m}$. Then f is holomorphic if and only if it is almost complex.*

Proof. A general element in $\mathbf{R}^{2n}$ will be denoted by $(x^1, \dots, x^n, y^1, \dots, y^n)$ and a general element in $\mathbf{R}^{2m}$ by $(u^1, \dots, u^m, v^1, \dots, v^m)$. The identifications we made are thus

$$(x^1,\dots,x^n,y^1,\dots,y^n)\in\mathbf{R}^{2n}\mapsto(x^1+\sqrt{-1}y^1,\dots,x^n+\sqrt{-1}y^n)\in\mathbf{C}^n,$$
$$(u^1,\dots,u^m,v^1,\dots,v^m)\in\mathbf{R}^{2m}\mapsto(u^1+\sqrt{-1}v^1,\dots,u^m+\sqrt{-1}v^m)\in\mathbf{C}^m.$$

Consider now f as a function of $U\subseteq\mathbf{C}^n\simeq\mathbf{R}^{2n}$ into $\mathbf{R}^{2m}$. We write $f=(u^1,\dots,u^m,v^1,\dots,v^m)$. Then f is holomorphic if and only if it satisfies the Cauchy-Riemann equations, that is

$$\frac{\partial u^k}{\partial x^j}-\frac{\partial v^k}{\partial y^j}=0,$$
$$\frac{\partial u^k}{\partial y^j}+\frac{\partial v^k}{\partial x^j}=0$$

for $j,k=1,\dots n$. On the other hand,

$$Df\left(\frac{\partial}{\partial x^j}\right)=\sum_{k=1}^{m}\frac{\partial u^k}{\partial x^j}\frac{\partial}{\partial u^k}+\frac{\partial v^k}{\partial x^j}\frac{\partial}{\partial v^k},$$
$$Df\left(\frac{\partial}{\partial y^j}\right)=\sum_{k=1}^{m}\frac{\partial u^k}{\partial y^j}\frac{\partial}{\partial u^k}+\frac{\partial v^k}{\partial y^j}\frac{\partial}{\partial v^k}$$

for $j=1,\dots,n$. By definition, $J(\partial/\partial x^j)=-\partial/\partial y^j$ and $J(\partial/\partial y^j)=\partial/\partial x^j$ for $j=1,\dots,n$, and $J'(\partial/\partial u^k)=-\partial/\partial v^k$ and $J'(\partial/\partial u^k)=\partial/\partial v^k$ for $k=1,\dots,m$. Using all the preceding relations, it is a simple matter of algebraic manipulation to show that the Cauchy-Riemann equations are satisfied if and only if $Df\circ J=J'\circ Df$. ■

Before continuing, we need to introduce some notation. Let M be again a C^∞ manifold. For $U\subseteq M$ open, we denote the space of $\mathbf{C}$-valued C^∞ functions by $C^\infty_{\mathbf{C}}(U)$. By $T^{\mathbf{C}}M$, we mean the complexification of the vector bundle TM. For $x\in M$, the fiber of $T^{\mathbf{C}}M$ at x is $T^{\mathbf{C}}_xM$, the complexification of T_xM. An element of $T^{\mathbf{C}}_xM$ is called a *(complex) vector at x*. Sections of $T^{\mathbf{C}}M$ are called *(complex) vector fields*. By $T^{*\mathbf{C}}M$ we mean the dualization of $T^{\mathbf{C}}M$, which has fiber $T^{*\mathbf{C}}_xM=(T_xM)^{*\mathbf{C}}$ at x. Elements of $\bigwedge^r T^{*\mathbf{C}}_xM$ are called *(complex) r-forms at x* and sections of $\bigwedge^r T^{*\mathbf{C}}M$ are called *(complex) (differential) r-forms*. We define $\Omega^r_{\mathbf{C}} U = C^\infty(U, \bigwedge^r T^{*\mathbf{C}} M)$ for $U \subseteq M$ open.

Remark 1.1.1. When discussing objects in the classic differential-geometric sense, we will sometimes use the adjective *real* for emphasis, so that we will say real vector fields, real differential forms, etc. This is justified since $T^{\mathbf{C}} M$ is the complexification of TM and $\bigwedge^r T^{*\mathbf{C}} M$ can be identified with the complexification of $\bigwedge^r T^* M$ in a canonical way.

Now, given an ACS J on M, there are a number of vector bundles that are natural to consider on M aside from the ones just introduced. First note that

$$T_x^{\mathbf{C}} M = T_x^{1,0} M \oplus T_x^{0,1} M,$$

where $T_x^{1,0} M$, respectively $T_x^{0,1} M$, is the $\sqrt{-1}$, respectively $-\sqrt{-1}$, eigenspace of J_x. Elements of $T_x^{1,0} M$, respectively $T_x^{0,1} M$, *tangent vectors of bidegree* $(1, 0)$, respectively $(0, 1)$, *at* x. The bundle $T^{1,0} M$, called the *holomorphic tangent bundle*, is the subbundle of $T^{\mathbf{C}} M$ with fiber $T_x^{1,0} M$ at x. Likewise, the bundle $T^{0,1} M$, called the *anti-holomorphic tangent bundle*, is the subbundle of $T^{\mathbf{C}} M$ with fiber $T_x^{0,1} M$ at x. Thus

$$T^{\mathbf{C}} M = T^{1,0} M \oplus T^{0,1} M. \tag{1.1.2}$$

Sections of $T^{1,0} M$, respectively $T^{0,1} M$, are called *(complex) vector fields of bidegree* $(1, 0)$, respectively $(0, 1)$. The bundle $\bigwedge^{p,q} T^{*\mathbf{C}} M$ is the subbundle of $\bigwedge^r T^{*\mathbf{C}} M$, $r = p + q$, with fiber $\bigwedge^{p,q} T_x^{*\mathbf{C}} M$ at x. Elements of $\bigwedge^{p,q} T_x^{*\mathbf{C}} M$ are called *(complex) r-forms of bidegree* (p, q) *at* x. Sections of $\bigwedge^{p,q} T^{*\mathbf{C}} M$ are called *(differential) forms of bidegree* (p, q) or *(differential)* (p, q)*-forms*. By Proposition 1.1.5,

$$\bigwedge^r T_x^{*\mathbf{C}} M = \bigoplus_{p+q=r} \bigwedge^{p,q} T_x^{*\mathbf{C}} M$$

and

$$\bigwedge^r T^{*\mathbf{C}} M = \bigoplus_{p+q=r} \bigwedge^{p,q} T^{*\mathbf{C}} M. \tag{1.1.3}$$

Furthermore, there is a natural isomorphism

$$\bigwedge^{p,q} T^{*\mathbf{C}}M \simeq \bigwedge^{p,0} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} \bigwedge^{0,q} T^{*\mathbf{C}}M. \tag{1.1.4}$$

Let $U \subseteq M$ be open. We define $\Omega^{p,q}U = C^\infty(U, \bigwedge^{p,q} T^{*\mathbf{C}}M)$ for $U \subseteq M$ open. Note that

$$\Omega^r_{\mathbf{C}}U = \bigoplus_{p+q=r} \Omega^{p,q}U.$$

Suppose (M', J') is another ACM and $f : M \to M'$ a C^∞ function. The differential $Df : TM \to TM'$ extends to a $C^\infty_{\mathbf{C}}(M)$-linear morphism of $T^{\mathbf{C}}M$ into $T^{\mathbf{C}}M'$, which we also denote by Df. We can make sense of a complex tangent vector acting on a complex-valued C^∞ function by extending by linearity the definition of a real tangent vector acting on a real-valued C^∞ function. Always using linearity, we extend the cxtcrior derivative d to a $\mathbf{C}$-linear operator on complex differential forms and the Lie bracket $[\cdot,\cdot]$ to a $\mathbf{C}$-bilinear operator on complex vector fields.

There is an important (real) tensor field of type $(1, 2)$ associated to the ACM (M, J), the so-called *Nijenhuis tensor*. This tensor field, which we denote by N^J, is defined by

$$N^J(X,Y) = 2\{[J(X), J(Y)] - [X,Y] - J([X, J(Y)]) - J([J(X), Y])\},$$

where X and Y are real vector fields over M. If $(x^1, \dots, x^{2n})$ are local coordinates on M, then the components $N^{J,i}_{jk}$ of N^J are given by

$$N^{J,i}_{jk} = 2\sum_h J^h_j \frac{\partial J^i_k}{\partial x^h} - J^h_k \frac{\partial J^i_j}{\partial x^h} - J^i_h \frac{\partial J^h_k}{\partial x^j} + J^i_h \frac{\partial J^h_j}{\partial x^k}. \tag{1.1.5}$$

Proposition 1.1.8. *The following statements are equivalent:*

(i) *The vector field* $[Z, W]$ *is of bidegree* $(1, 0)$ *whenever* $Z, W \in C^\infty(M, T^{\mathbf{C}}M)$ *are.*

(ii) *The vector field* $[Z, W]$ *is of bidegree* $(0, 1)$ *whenever* $Z, W \in C^\infty(M, T^{\mathbf{C}}M)$ *are.*

(iii) *We have* $d\Omega^{1,0}M \subseteq \Omega^{2,0}M \oplus \Omega^{1,1}M$ *and* $d\Omega^{0,1}M \subseteq \Omega^{1,1}M \oplus \Omega^{0,2}M$ *.*

(iv) *We have* $d\Omega^{p,q}M \subseteq \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$ *for* $p, q = 0, 1, \ldots, n$*.*

(v) *The Nijenhuis tensor* N^J *vanishes.*

Proof.

(i) $\Longleftrightarrow$ **(ii)** This follows from the fact that $\overline{[Z, W]} = [\overline{Z}, \overline{W}]$ and a vector field is of bidegree (0, 1) if and only if its conjugate is of bidegree (1, 0). (Proposition 1.1.3).

(ii) $\Longleftrightarrow$ **(iii)** Suppose (ii), and hence (i), holds. Let $\omega \in \Omega^{1,0}M$ and $Z, W \in C^\infty(M, T^{0,1}M)$. Using (Kobayashi et Nomizu, 1963, Proposition 3.11, ch. 1), we have

$$d\omega(Z, W) = \frac{1}{2}\{Z(\omega(W)) - W(\omega(Z)) - \omega([Z, W])\}.$$

But since $\omega(V) = 0$ for $V \in C^\infty(M, T^{0,1}M)$ (see Proposition 1.1.4), the above vanishes. Now $d\omega \in \Omega^2_{\mathbb{C}}M = \Omega^{2,0}M \oplus \Omega^{1,1}M \oplus \Omega^{0,2}$. Let η be the $\Omega^{0,2}M$ component of $d\omega$. We have $d\omega(Z, W) = \eta(Z, W) = 0$. By retracing the definitions, we see that $\eta = 0$ and thus $d\omega \in \Omega^{2,0}M \oplus \Omega^{1,1}M$. That $d\Omega^{0,1}M \subseteq \Omega^{1,1}M \oplus \Omega^{0,2}M$ is proved in the same way, mutatis mutandis, using (i).

Conversely, suppose (iii) holds. Fix $Z, W \in C^\infty(M, T^{0,1}M)$ and let $\omega \in \Omega^{1,0}M$. Since $d\omega$ has no $\Omega^{0,2}M$ component, then

$$d\omega(Z, W) = -\frac{1}{2}\omega([Z, W]) = 0.$$

As this holds true for all $\omega \in \Omega^{1,0}M$, $[Z, W]$ must be of bidegree (0, 1) (see Proposition 1.1.4).

(iii) $\Longleftrightarrow$ **(iv)** Suppose (iii) holds. Let $U \subseteq M$ be an open subset sufficiently small to admit local frames for $\bigwedge^{1,0} T^{*\mathbb{C}}M$ and $\bigwedge^{0,1} T^{*\mathbb{C}}M$, say $\{\eta^1, \ldots, \eta^n\}$ and $\{\xi^1, \ldots, \xi^n\}$ respectively. Then it should be clear that given $\omega \in \Omega^{p,q}M$, one has

$$\omega = f \wedge \eta^{i_1} \wedge \cdots \wedge \eta^{i_p} \wedge \xi^{j_1} \wedge \cdots \wedge \xi^{j_q}$$

in U, where $f \in C_c^\infty(U)$ and $1 \le i_1 < i_2 < \ldots < i_p \le n$, $1 \le j_1 < j_2 < \ldots < j_q \le n$. Since $df \in \Omega^{1,0}U \oplus \Omega^{0,1}U$, $d\eta_i \in \Omega^{2,0}U \oplus \Omega^{1,1}U$ and $d\xi_j \in \Omega^{1,1}U \oplus \Omega^{0,2}U$, (iv) holds. That (iv) implies (iii) is trivial.

(ii) $\iff$ (v) Let $Z, W \in C^\infty(M, T^{0,1}M)$. There are $X, Y \in C^\infty(M, TM)$ such that $Z = X + \sqrt{-1}J(X)$ and $W = Y + \sqrt{-1}J(Y)$ (Proposition 1.1.3). Putting $V = [Z, W]$, the equality

$$V - \sqrt{-1}J(V) = N^J(X, Y) - \sqrt{-1}J(N^J(X, Y))$$

can be verified by developing both sides. Multiplying left and right by $\frac{1}{2}$, the LHS becomes the projection of V onto $T^{1,0}M$ (this can be deduced from Proposition 1.1.3). It is then seen at once that $V \in C^\infty(M, T^{0,1}M)$ if and only if $N^J(X, Y)$ vanishes everywhere.

■

An ACS J for which the equivalent statements in the above proposition hold is said to be *formally integrable*. In the next section, we will focus exclusively on formally integrable ACS's. As the example below shows, not all ACS's are formally integrable.

Example 1.1.2. We exhibit an ACS on the 6-sphere S^6 which is not formally integrable. It is constructed via the algebra of octonions. This is the algebra over $\mathbf{R}$ one obtains by endowing the vector space $\mathbf{O} = \mathbf{R}\langle 1, e_0, \ldots, e_6\rangle$ with the product defined by the following multiplication table, 1 being the multiplicative identity:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6
e_0	-1	e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$
e_1	$-e_2$	-1	e_0	$-e_5$	e_6	e_3	$-e_4$
e_2	e_1	$-e_0$	-1	e_6	e_5	$-e_4$	$-e_3$
e_3	$-e_4$	e_5	$-e_6$	-1	e_0	$-e_1$	e_2
e_4	e_3	$-e_6$	$-e_5$	$-e_0$	-1	e_2	e_1
e_5	$-e_6$	$-e_3$	e_4	e_1	$-e_2$	-1	e_0
e_6	e_5	e_4	e_3	$-e_2$	$-e_1$	$-e_0$	-1

.

This algebra is neither commutative nor associative, however it is alternative, meaning that for $x, y, z \in \mathbf{O}$, one has $(x \cdot x) \cdot y = x \cdot (x \cdot y)$ and $x \cdot (y \cdot y) = (x \cdot y) \cdot y$. The *real part* of an octonion is its projection onto $\mathbf{R}\langle 1 \rangle$ and its *imaginery part* is its projection onto $\mathbf{R}\langle e_0, \ldots, e_6 \rangle$. A *real*, respectively *pure imaginery*, octonion is one whose imaginary, respectively real, part is 0. The space of real octonions naturally identifies with $\mathbf{R}$ and we may thus regard $\mathbf{O}$ as an extension of $\mathbf{R}$. The purely imaginary octonions form a 7-dimensional real vector space denoted by $\mathcal{Y}$. We equip $\mathcal{Y}$ with a product $\times$ by defining $a \times b$ to be the imaginary part of the octonion product $a \cdot b$. If $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^7 \simeq \mathcal{Y}$ then one has

$$x \times y - \langle x, y \rangle = x \cdot y,$$

$$\langle x \times y, z \rangle = \langle x, y \times z \rangle,$$

$$x \cdot x = -\langle x, x \rangle,$$

$$x \times x = 0,$$

for $x, y, z \in \mathcal{Y}$. We thus see that $x \times y$ is orthogonal to both x and y. Furthermore, if x is orthogonal to y then $x \times y = x \cdot y$. Consider now S^6 as the locus of the equation $\langle x, x \rangle = 1$, $x \in \mathcal{Y}$. Given $x \in S^6$, let J_x be the map $y \mapsto x \times y$. If $y \in T_x S^6$, then because $x \perp y$ and $x \times y \perp x$, we have

$$J_x^2(y) = x \times (x \times y) = x \cdot (x \cdot y) = (x \cdot x) \cdot y = -y.$$

Thus the assignment $x \mapsto J_x$ gives an ACS J on S^6.

We will show that the Nijenhuis tensor of J does not vanish everywhere. Writing an arbitrary element x of $\mathcal{Y} \simeq \mathbf{R}^7$ as $(X^0, \ldots, X^6)$, we put on S^6 the local coordinates $\{x^i = X^i \ : \ i = 1, \ldots, 6\}$, which are defined on $U = \{x \in S^6 \ : \ X^0 > 0\}$. For $x \in U$, let A^x denote the matrix of J_x, seen as a linear map from $\mathcal{Y}$ to itself, with respect to the basis $\{e_0, \ldots, e_6\}$. By referring to the multiplication table given above, we readily compute

$$A^x = \begin{bmatrix} 0 & -X^2 & X^1 & -X^4 & X^3 & -X^6 & X^5 \\ X^2 & 0 & -X^0 & X^5 & -X^6 & -X^3 & X^4 \\ -X^1 & X^0 & 0 & -X^6 & -X^5 & X^4 & X^3 \\ X^4 & -X^5 & X^6 & 0 & -X^0 & X^1 & -X^2 \\ -X^3 & X^6 & X^5 & X^0 & 0 & -X^2 & -X^1 \\ X^6 & X^3 & -X^4 & -X^1 & X^2 & 0 & -X^0 \\ -X^5 & -X^4 & -X^3 & X^2 & X^1 & X^0 & 0 \end{bmatrix}. \tag{1.1.6}$$

Let a^x denote the matrix of J_x, this time seen as a linear map from $T_x S^6$ to itself, with respect to $\{\partial/\partial x^1|_x, \ldots, \partial/\partial x^6|_x\}$. It is convenient to index the rows and columns of A^x by $0, \ldots, 6$. It is then easy to see that for $1 \le i, j \le 6$,

$$a^x_{ij} = A^x_{ij} - A^x_{i0}\frac{X^j}{X^0}. \tag{1.1.7}$$

We thus have

$$\frac{\partial a^x_{ij}}{\partial x^k} = \frac{\partial A^x_{ij}}{\partial x^k} - \frac{A^x_{i0}}{\partial x^k} \cdot \frac{X^j}{X^0} - A^x_{i0} \cdot \frac{\frac{\partial X^j}{\partial x^k} \cdot X^0 - X^j \cdot \frac{\partial X^0}{\partial x^k}}{X^0 \cdot X^0}.$$

At the point $e_1 = (1, 0, \ldots, 0)$, one has $\partial X^0/\partial x^k = 0$ for $k = 1, \ldots, 6$. Furthermore, from (1.1.6), we have $A^{e_1}_{i0} = 0$. It follows that

$$\left.\frac{\partial a^x_{ij}}{\partial x^k}\right|_{e_1} = \left.\frac{\partial A^x_{ij}}{\partial x^k}\right|_{e_1} = \left.\frac{\partial A^x_{ij}}{\partial X^k}\right|_{e_1}.$$

Also, because of (1.1.7), $a_{ij}^{e_1} = A_{ij}^{e_1}$. As a result, following (1.1.5) and referring to (1.1.6),

$$\begin{aligned} N_{15}^{J,4}\Big|_{e_1} &= 2\left(a_{21}^{e_1}\cdot\frac{\partial a_{45}^{x}}{\partial x^2}\Big|_{e_1} - a_{65}^{e_1}\cdot\frac{\partial a_{41}^{x}}{\partial x^6}\Big|_{e_1} - a_{43}^{e_1}\cdot\frac{\partial a_{35}^{x}}{\partial x^1}\Big|_{e_1} + a_{43}^{e_1}\cdot\frac{\partial a_{31}^{x}}{\partial x^5}\Big|_{e_1}\right) \\ &= 2\left(A_{21}^{e_1}\cdot\frac{\partial A_{45}^{x}}{\partial X^2}\Big|_{e_1} - A_{65}^{e_1}\cdot\frac{\partial A_{41}^{x}}{\partial X^6}\Big|_{e_1} - A_{43}^{e_1}\cdot\frac{\partial A_{35}^{x}}{\partial X^1}\Big|_{e_1} + A_{43}^{e_1}\cdot\frac{\partial A_{31}^{x}}{\partial X^5}\Big|_{e_1}\right) \\ &= -2. \end{aligned}$$

Thus N^J does not vanish at e_1 and so J is not formally integrable.

1.2 Complex manifolds

1.2.1 Basic definitions

Among ACM's, we will be particularly interested in a special subclass, that is complex manifolds.

Definition 1.2.1. A *complex manifold* is an ACM (M, J) for which J is formally integrable.

There is another, equivalent, way of defining complex manifolds. This alternative definition simply mimics the standard definition of a C^∞ manifold, replacing diffeomorphic transition maps by biholomorphic ones. If M is a Hausdorff and paracompact space, an atlas for M consisting of charts to open subsets of $\mathbf{C}^n$ such that the transition maps are all biholomorphic is called a *holomorphic atlas (for M)*.

Definition 1.2.2. A *complex manifold* of dimension n is a Hausdorff and paracompact space M together with a maximal holomorphic atlas for M, that is a holomorphic atlas not contained in any holomorphic atlas other than itself.

Given a complex manifold M of dimension n as per the above definition, a chart contained in the associated maximal holomorphic atlas is called a *holomorphic chart*.

Evidently, M harbours an underlying C^∞ manifold of dimension $2n$, which we also denote by M. To distinguish between the dimension of M as a complex manifold (as per Definition 1.2.2) and its dimension as a C^∞ manifold, we will call the former its *complex dimension* and the second its *real dimension*. Given two complex manifolds M and M' as per Definition 1.2.2, a map f of an open subset of M into M' is said to be *holomorphic*, *complex analytic*, or a *holomorphism* if each of its coordinate representations is holomorphic. If f is a bijection between an open subset of M and an open subset of M' and both it and its inverse are holomorphic, then it is said to be *biholomorphic* or a *biholomorphism*. Note that a holomorphic map, considered as a map between C^∞ manifolds, is smooth.

It is easy to show that Definition 1.2.1 subsumes Definition 1.2.2. Indeed, let M be a complex manifold (as per Definition 1.2.2) of complex dimension n. We identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ as in Example 1.1.1. Let $\varphi : U \subseteq M \to \varphi(U) \subseteq \mathbf{C}^n$ be a holomorphic chart. Seeing φ as a C^∞ chart into $\mathbf{R}^{2n}$, $D\varphi^{-1}\big|_x \circ J \circ D\varphi|_x$, where J is the standard ACS on $\mathbf{R}^{2n}$ (Example 1.1.1), gives us an ACS on T_xM for any $x \in U$. This ACS is in fact independent of the chart φ. To see this, suppose that $\psi : V \subseteq M \to \psi(V)$ is another coordinate chart and $x \in U \cap V$. The ACS's these charts give us on T_xM are $D\varphi^{-1}\big|_x \circ J \circ D\varphi|_x$ and $D\psi^{-1}\big|_x \circ J \circ D\psi|_x$ respectively. But since the transition map $\psi \circ \varphi^{-1}$ is holomorphic, we have

$$D(\psi \circ \varphi^{-1})\big|_x \circ J = J \circ D(\psi \circ \varphi^{-1})\big|_x .$$

by Proposition 1.1.7. Using the chain rule, we get $D\varphi^{-1}\big|_x \circ J \circ D\varphi|_x = D\psi^{-1}\big|_x \circ J \circ D\psi|_x$. By assigning to every $x \in M$ an ACS on T_xM as we just did, we define unambiguously an ACS on the whole manifold M, which we also denote by J. Computing the Nijenhuis tensor N^J, we see that it vanishes and thus (M, J) is a complex manifold as per Definition 1.2.1.

An ACS such as J, that is an ACS on a C^∞ manifold M that is induced in the above manner by a complex manifold (as per Definition 1.2.2) of which M is the underlying C^∞ manifold is said to be *integrable* or simply a *complex structure*. As we have just seen, if an ACS is integrable, then it is formally integrable. The converse is also true, but much more difficult to show. This is the content of the Newlander-Nirenberg theorem, which is proven in Appendix A.

Now let M' be another complex manifold (as per Definition 1.2.2) and J' its corresponding integrable ACS. An easy corollary of Proposition 1.1.7 is

Proposition 1.2.1. *Let f be a function of an open subset of M into M'. Then f is holomorphic if and only if it is almost complex with respect to the ACM's (M, J) and (M', J').* ■

From now on, we will use definitions 1.2.1 and 1.2.2 simultaneously and only talk of integrable (as opposed to formally integrable) ACS's. The following proposition is proven in exactly the same way as its C^∞ counterpart.

Proposition 1.2.2. *Suppose that G is a group acting on a complex manifold M freely, properly discontinuously and by biholomorphisms. Then there is a unique way to turn M/G into a complex manifold so that the projection map $\pi : M \to M/G$ is a local biholomorphism (i.e. around each $x \in M$ there is an open neighbourhood which is mapped biholomorphically onto an open subset of M/G by π).* ■

We now give a few examples of complex manifolds.

Example 1.2.1 (Euclidean space). The simplest complex manifolds are the complex Euclidean spaces $\mathbf{C}^n$.

Example 1.2.2 (Complex projective space). The complex projective space $\mathbf{CP}^n$ is the space $\mathbf{C}^{n+1}\backslash\{0\}/\sim$, where $\sim$ is the equivalence relation obtained by putting $x \sim y$ if and only if there is a nonzero complex number α such that $\alpha x = y$. An equivalence class in $\mathbf{CP}^n$ with representative $(z^1, \ldots, z^{n+1})$ is denoted by $[z^1 : \cdots : z^{n+1}]$. We define

$$U_i = \{[z^1 : \cdots : z^{n+1}] \ : \ z^i \neq 0\}$$

and let $\varphi_i : U_i \to \mathbf{C}^n$ be the map which sends $[z^1 : \cdots : z^{n+1}]$ to the n-tuple obtained by first deleting z^i from $(z^1, \ldots, z^n)$ and then normalizing by z^i (this is well-defined). Then as is easily verified $\{(U_i, \varphi_i)\}_{1 \leq i \leq n+1}$ is a holomorphic atlas for $\mathbf{CP}^n$.

Example 1.2.3 (Riemann surfaces). Suppose M is a 2-dimensional C^∞ manifold and that J is an ACS on M. Then $\bigwedge^{2,0} T^{*\mathbf{C}}M$ and $\bigwedge^{0,2} T^{*\mathbf{C}}M$ are both trivial. Thus item (iii) of Proposition 1.1.8 is satisfied and J is integrable. So (M, J) is a complex manifold, called a *Riemann surface*.

Example 1.2.4 (Blow-ups). Suppose we are given the following data:

(i) a complex manifold M and a compact complex submanifold $S \subseteq M$;

(ii) open neighbourhoods W and W_1 with $S \subseteq W_1 \Subset W \subseteq M$;

(iii) a complex manifold $\tilde{W}$, a compact complex submanifold $\tilde{S}$ of $\tilde{W}$ and an open neighbourhood $\tilde{W}_1$ of $\tilde{S}$ with $\tilde{S} \subseteq \tilde{W}_1 \Subset \tilde{W}$;

(iv) a biholomorphism $\Phi : \tilde{W} - \tilde{S} \to W - S$ such that $\Phi(\tilde{W}_1 - \tilde{S}) = W_1 - S$.

One can then perform "surgery" to replace the submanifold S by $\tilde{S}$ in the manifold M. More precisely, one defines

$$(M - S) \cup \tilde{S} = \frac{(M - \overline{W_1}) \sqcup \tilde{W}}{\sim}$$

where $\sim$ identifies $p \in \tilde{W} - \overline{\tilde{W}_1}$ with $\Phi(p) \in M - \overline{W_1}$. It is then easily shown that $(M - S) \cup \tilde{S}$ is naturally a complex manifold.

If M is of dimension n and $p \in M$, there is a standard method to define the manifold $(M - \{p\}) \cup \mathbf{CP}^{n-1}$. The procedure is called *blowing up* M at p and the manifold $(M - \{p\}) \cup \mathbf{CP}^{n-1}$ is called the *blow-up* of M at p.

1.2.2 Dolbeaut operators and holomorphic objects

Let (M, J) be a complex manifold of real dimension $2n$. We have $d\Omega^{p,q}M \subseteq \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$ for $p, q \geq 0$. The composition of the projection of $\Omega^{p,q}M$ onto $\Omega^{p+1,q}M$, respectively $\Omega^{p,q}M$, and d is denoted by ∂, respectively $\overline{\partial}$. These two operators are called the *Dolbeaut operators*. By Proposition 1.1.8, $d\omega = \partial\omega + \overline{\partial}\,\omega$ for $\omega \in \Omega^{p,q}M$. Since $d^2 = 0$, one has

$$\partial^2 = 0,$$
$$\overline{\partial}^2 = 0,$$
$$\partial\,\overline{\partial} + \overline{\partial}\,\partial = 0.$$

For $\Lambda = \partial$ or $\Lambda = \overline{\partial}$, the Leibniz rule

$$\Lambda(\alpha \wedge \beta) = (\Lambda\alpha) \wedge \beta + (-1)^k \alpha \wedge \Lambda\beta \tag{1.2.1}$$

is satisfied, where α and β are differential forms, α being of degree k. By convention, the operators ∂ and $\overline{\partial}$ are simply interpreted as the 0 operator when acting on $\Omega^{p,q}M$ with $p < 0$ or $q < 0$.

We have already defined what it means for a C^∞ function sending an open subset of M into another complex manifold to be holomorphic. The following proposition states that, in the case of a $\mathbf{C}$-valued function, there is a condition involving the operator $\overline{\partial}$ that is equivalent to holomorphicity.

Proposition 1.2.3. *Let f be a C^∞ $\mathbf{C}$-valued function defined on an open subset of M. Then f is holomorphic if and only if $\overline{\partial} f = 0$.*

Proof. We write $f = u + \sqrt{-1}v$ where u and v are C^∞ $\mathbf{R}$-valued functions. Let J' be the standard ACS on $\mathbf{R}^2 \simeq \mathbf{C}$. Consider f as an $\mathbf{R}^2$-valued function for the moment. By Proposition 1.2.1, f is holomorphic if and only if $Df|_x \circ J_x = J' \circ Df|_x$ for every

x in the domain of f. This last condition just means that if $X \in T_x M$, then

$$Du|_x(J(X)) = -Dv|_x(X), \qquad (1.2.2)$$
$$Dv|_x(J(X)) = Du|_x(X).$$

Consider f as a **C**-valued function now, and $Df|_x$ as a functional on $T_x^{\mathbf{C}} M$. Then

$$\begin{aligned} Df|_x(X + \sqrt{-1}J(X)) &= (Du|_x + \sqrt{-1}\, Dv|_x)(X + \sqrt{-1}J(X)) \\ &= (Du|_x(X) - Dv|_x(J(X))) + \\ &\quad \sqrt{-1}(Dv|_x(X) + Du|_x(J(X))) \end{aligned}$$

and we see that the above vanishes if and only if (1.2.2) holds. Since $X + \sqrt{-1}J(X)$ is just an arbitrary tangent vector of bidegree (0, 1) at x, we conclude in view of Proposition 1.1.4 that f is holomorphic if and only if df is of bidegree $(1, 0)$, that is $\bar{\partial} f = 0$. ■

A complex vector field X of bidegree $(1, 0)$ is said to be *holomorphic* if Xf is holomorphic for every **C**-valued holomorphic function defined on an open subset of M. In turn, a differential form ω of bidegree $(p, 0)$ is said to be holomorphic if $\omega(X_1, \ldots, X_p)$ is holomorphic for every holomorphic vector fields $X_1, \ldots, X_p$.

1.2.3 In local coordinates

Let M be a complex manifold of complex dimension n. Let $\varphi : U \subseteq M \to \mathbf{C}^n$ be a holomorphic chart on M. There are both real and complex coordinates induced by φ on U, which we denote by $x^1, \ldots, x^n, y^1, \ldots, y^n$ and $z^1, \ldots, z^n$ respectively. We have $z^i = x^i + \sqrt{-1}y^i$, $i = 1, \ldots, n$. It is customary to write $\bar{z}^i$ instead of $\overline{z^i}$. We see that $dz^i = dx^i + \sqrt{-1}dy^i$ and $d\bar{z}^i = dx^i - \sqrt{-1}dy^i$, which are forms in $\Omega^{1,0}U$ and $\Omega^{0,1}U$ respectively. The coordinates $z^1, \ldots, z^n$ are called *holomorphic*. We define

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right),$$
$$\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right).$$

By item (i) of Proposition 1.1.3, $\partial/\partial z^1, \dots, \partial/\partial z^n$ and $\partial/\partial \overline{z}^1, \dots, \partial/\partial \overline{z}^n$ are local frames for $T^{1,0}M$ and $T^{0,1}M$. The duals of those frames are $dz^1, \dots, dz^n$ and $d\overline{z}^1, \dots, d\overline{z}^n$ and so form frames for $\bigwedge^{1,0} T^{*\mathbf{C}}M$ and $\bigwedge^{0,1} T^{*\mathbf{C}}M$ respectively. Thus a local frame for $\bigwedge^{p,q} T^{*\mathbf{C}}M$, $p, q = 0, 1, \dots, n$, is

$$\{dz^{j_1} \wedge \dots \wedge dz^{j_p} \wedge d\overline{z}^{k_1} \wedge \dots \wedge d\overline{z}^{k_q} \ : \ 1 \leq j_1 < j_2 < \dots < j_p \leq n\ ,$$
$$1 \leq k_1 < k_2 < \dots < k_q \leq n\}.$$

Let $f \in C^\infty_{\mathbf{C}}(U)$. We have

$$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z^i} dz^i,$$
$$\overline{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \overline{z}^i} d\overline{z}^i.$$

Thus by Proposition 1.2.3, f is holomorphic if and only if $\partial f / \partial \overline{z}^i = 0$ for $i = 1, \dots, n$. Let X be a vector field of bidegree $(1, 0)$ over U. We write

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial \overline{z}^i},$$

from which we see that X is holomorphic if and only if the component functions X^i, $i = 1, \dots, n$, are holomorphic. Likewise, a differential form of bidegree $(p, 0)$, $p = 0, 1, \dots, n$, is holomorphic if and only if its component functions with respect to the frame $\{dz^{j_1} \wedge \dots \wedge dz^{j_p} \ : \ 1 \leq j_1 < j_2 < \dots < j_p \leq n\}$ are holomorphic. We thus find

Proposition 1.2.4. *Let ω be a differential $(p, 0)$-form, $p = 0, 1, \dots, n$. Then ω is holomorphic if and only if $\overline{\partial}\, \omega = 0$.* ■

1.2.4 Holomorphic vector bundles

Let M be a complex manifold. By replacing diffeomorphic maps by holomorphic ones in the definition of a smooth vector bundle, we obtain the definition of a *holomorphic* *vector bundle*. Thus a holomorphic vector bundle E of rank r over M is a complex manifold together with a holomorphic projection map $\pi : E \to M$ such that for each $p \in M$ 1. the fiber $\pi^{-1}(\{p\})$ has a complex vector space structure, 2. there exists an open neighbourhood $U \subseteq M$ of p and a biholomorphic map $\phi : \pi^{-1}(U) \to U \times \mathbf{C}^r$ which commutes with the projections onto U, 3. the fiberwise restrictions of ϕ are $\mathbf{C}$-linear.

Note that if a holomorphic vector bundle E is considered as a smooth manifold then it is simply a complex vector bundle over M. Conversely, if E is a complex vector bundle over M with projection map π and $\{(U_i, \psi_i)\}_{i\in I}$ is a trivializing cover of M, then if the transition maps $\psi_i \circ \psi_j^{-1}$, $i, j \in I$, are holomorphic, E can be naturally turned into a holomorphic vector bundle over M.[1]

Just like in the case of a smooth vector bundles, new holomorphic bundles can be constructed from old ones by applying canonical algebraic constructions fiberwise. Thus duals, exterior powers, direct sums and tensor products of holomorphic vector bundles are well-defined holomorphic vector bundles in their own right.

Example 1.2.5. Let M be a n-dimensional complex manifold. Consider the holomorphic tangent bundle $T^{1,0}M$ and denote the projection map by π. As its name suggests, $T^{1,0}M$ can be regarded as a holomorphic vector bundle. Indeed, let $\{(U_i, \varphi_i)\}_{i\in I}$ be a holomorphic atlas for M. Denoting the holomorphic coordinates thus defined on U_i by $(z_i^1, \dots, z_i^n)$, let ψ_i be the trivialization of $T^{1,0}M$ over U_i which induces the local frame $\{\frac{\partial}{\partial z_i^1}, \cdots, \frac{\partial}{\partial z_i^n}\}$. Then $\{(U_i, \phi_i)\}_{i\in I}$ is a trivializing cover of M for $T^{1,0}M$ and, as is easily seen, the transition maps $\phi_i \circ \phi_j^{-1}$, $i, j \in I$, are holomorphic. Thus $T^{1,0}M$ can be turned into a holomorphic vector bundle. This is canonical since, quite obviously, it does not depend on the holomorphic atlas with which we started. Note that the holomorphic

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[1] Starting with different trivializing covers may, however, result in non-isomorphic holomorphic vector bundles.

sections of $T^{1,0}M$ are exactly the holomorphic vector fields over M.

Example 1.2.6. Since $\bigwedge^{1,0} T^{*\mathbf{C}}M$ can be canonically identified with the dual of $T^{1,0}M$, it follows that it is also a holomorphic vector bundle. This in turn implies that $\bigwedge^{p,0} T^{*\mathbf{C}}M$ is a holomorphic vector bundle. The holomorphic sections of $\bigwedge^{p,0} T^{*\mathbf{C}}M$ are exactly the holomorphic $(p, 0)$-forms.

Suppose E is a holomorphic vector bundle over M of rank r. Seeing E as a complex vector bundle, consider the complex vector bundle

$$\bigwedge^{p,q} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} E.$$

The space of its smooth sections, i.e. smooth E-valued (p, q)-forms, is denoted by $\mathcal{A}^{p,q}E$, or simply $\mathcal{A}^q E$ when $p = 0$. We can define a Dolbeaut operator $\overline{\partial} : \mathcal{A}^{p,q}E \to \mathcal{A}^{p,q+1}E$. To do this, let $e^1, \dots, e^r$ be a frame consisting of holomorphic sections of E over an open subset $U \subseteq M$. Given $\alpha \in \mathcal{A}^{p,q}E$, we can write it as

$$\sum_i^r \alpha^r \otimes e^r,$$

in U, where $\alpha^1, \dots, \alpha^r \in \Omega^{p,q}U$. We then define

$$\overline{\partial}\alpha\Big|_U = \sum_i^r (\overline{\partial}\alpha^r) \otimes e^r,$$

where $\overline{\partial}$ as it appears on the right hand side is the usual Dolbeaut operator on $\Omega^{p,q}M$ previously introduced. It is an easy matter to verify that this definition is unambiguous and that $\overline{\partial}^2 = 0$. Moreover $\overline{\partial}$ satisfies the Leibniz rule

$$\overline{\partial}(f\alpha) = \overline{\partial} f \wedge \alpha + f \wedge \overline{\partial}\alpha \tag{1.2.3}$$

for $f \in C^\infty_{\mathbf{C}}(M)$ and $\alpha \in \mathcal{A}^{p,q}E$. Note that when $E = M \times \mathbf{C}$, then $\mathcal{A}^{p,q}E = \Omega^{p,q}M$ and this definition of $\overline{\partial}$ agrees with the previous one.

1.3 Dolbeaut cohomology

Dolbeaut cohomology is a complex-geometric analogue of DeRham cohomology. Suppose M is a complex manifold and E is a holomorphic vector bundle over M. Let $\overline{\partial}_{p,q}$ denote the Dolbeaut operator $\overline{\partial}$ on $\mathcal{A}^{p,q}E$. Fix $p \geq 0$. Since $\overline{\partial}^2 = 0$, the collection $\{(\mathcal{A}^{p,q}E, \overline{\partial}_{p,q}) : -\infty < q < \infty\}$ forms a differential complex. The q^{th} cohomology group of this complex is denoted by $H^{p,q}_{\overline{\partial}}(M, E)$, i.e. $H^{p,q}_{\overline{\partial}}(M, E) = \ker \overline{\partial}_{p,q} \,/\, \operatorname{im} \overline{\partial}_{p,q-1}$. We will see that, for $q \geq 1$, each of these groups is isomorphic to a Čech cohomology group of a certain sheaf.

Let F be a (real or complex) vector bundle over M. Given a C^∞ section ω of F and a point $x \in M$ in the domain of ω, $[\omega]_x$ denotes the germ at x of ω. We can speak of the sheaf of germs of C^∞ sections of F, $\mathscr{F}$. The stalk at $x \in M$ of this sheaf is the space $\mathscr{F}_x = \{[\omega]_x : \omega \text{ is a } C^\infty \text{ section of } F \text{ defined around } x\}$. The collection of sets $\mathcal{U}(\omega, U) = \{[\omega]_x : x \in U\}$, where ω is C^∞ section of F and U an open subset of its domain, forms a basis for the topology of $\mathscr{F}$.

We denote by $\mathscr{A}^{p,q}E$ the sheaf of germs of C^∞ E-valued (p,q)-forms. There is an important subsheaf of $\mathscr{A}^{p,0}E$, the sheaf of holomorphic E-valued $(p,0)$-forms, which we denote by $\Theta^p E$. This sheaf's stalk at $x \in M$ is

$$\{[\omega]_x : \omega \text{ is a holomorphic section of } \textstyle\bigwedge^{p,0} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} E \text{ defined around } x\}.$$

The Dolbeaut operator $\overline{\partial}$ induces a morphism of sheaves, also denoted by $\overline{\partial}$, taking $\mathscr{A}^{p,q}E$ to $\mathscr{A}^{p,q+1}E$ with $\overline{\partial}[\omega]_x = [\overline{\partial}\,\omega]_x$. The mapping taking $\Gamma(M, \mathscr{A}^{p,q}E)$ into $\Gamma(M, \mathscr{A}^{p,q+1}E)$ by sending ξ to the element in $\Gamma(M, \overline{\partial}\,\mathscr{A}^{p,q}E)$ whose value at $x \in M$ is $\overline{\partial}\,\xi(x)$ is also denoted by $\overline{\partial}$.

Before continuing, we need the following lemma, which is analogous to the Poincaré Lemma. We do not provide a proof, but the reader can find one in (Gunning et Rossi, 2009).

Theorem 1.3.1 (Dolbeaut lemma). *Let $U \subseteq \mathbf{C}^n$ be a bounded polydisc in complex Euclidean space. Suppose V is an open neighbourhood of $\overline{U}$ and $\omega \in \Omega^{p,q}V$, where $q \geq 1$, is a differential form such that $\overline{\partial}\,\omega = 0$. Then there exists a differential form $\eta \in \Omega^{p,q-1}U$ such that $\overline{\partial}\,\eta = \omega|_U$.* ■

Now from Proposition 1.2.4, we see that $\Theta^p E$ is in fact the kernel of $\overline{\partial} : \mathscr{A}^{p,0}E \to \mathscr{A}^{p,1}E$. Together with the Dolbeaut lemma, this means that

$$0 \to \Theta^p E \xrightarrow{\iota} \mathscr{A}^{p,0}E \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1}E \xrightarrow{\overline{\partial}} \mathscr{A}^{p,2}E \to \dots \tag{1.3.1}$$

is a long exact sequence of sheaves, in other words a so-called resolution of $\Theta^p E$. This resolution is fine, i.e. the sheaves $\mathscr{A}^{p,1}E, \mathscr{A}^{p,2}E, \dots$ are fine.[2]

By a standard result in the theory of sheaf cohomology, we have

$$H^q(M, \Theta^p E) \simeq \frac{\Gamma(M, \overline{\partial}\,\mathscr{A}^{p,q-1}E)}{\overline{\partial}\,\Gamma(M, \mathscr{A}^{p,q-1}E)}, \quad q \geq 1.$$

Since there is a natural isomorphism

$$\frac{\ker \overline{\partial}_{p,q}}{\operatorname{im} \overline{\partial}_{p,q-1}} \simeq \frac{\Gamma(M, \overline{\partial}\,\mathscr{A}^{p,q-1}E)}{\overline{\partial}\,\Gamma(M, \mathscr{A}^{p,q-1}E)},$$

we conclude

Theorem 1.3.2 (Dolbeaut theorem). *We have*

$$H^q(M, \Theta^p E) \simeq H^{p,q}_{\overline{\partial}}(M, E), \quad q \geq 1.$$

■

[2]A sheaf $\mathscr{L}$ is said to be fine if it admits partitions of unity, that is for any cover $\{U_i\}$ by open sets of the base manifold M there is a family of sheaf morphisms $\{\eta_i : \mathscr{L} \to \mathscr{L}\}$ such that 1. the support of η_i is compactly contained in U_i and 2. $\sum_i \eta_i$ is the identity $\mathrm{Id} : \mathscr{L} \to \mathscr{L}$.

CHAPTER II

ANALYTIC PRELIMINARIES

In this chapter we will quickly go over some analytic notions that will be needed later. Proofs are not given, the reader is referred to (Besse, 1987), (Narasimhan, 2001), (Nicolaescu, 2014) and (Wells, 2008) for more details. Throughout this chapter, M is a compact n-dimensional C^∞ manifold.

2.1 Norms on vector bundles

In elementary real analysis one encounters several norms of functions between Euclidean spaces. Typically, given such a norm $||\cdot||$ and positive integers m and n, one has for each bounded open subset $U \subseteq \mathbf{R}^n$ a specified space of $\mathbf{C}^m$-valued functions $\mathcal{V}(U)$ on which $||\cdot||$ is defined, with $(\mathcal{V}(U), ||\cdot||)$ being Banach.

Example 2.1.1. For the C^k norm $||\cdot||_{C^k}$, $k \geq 0$, $\mathcal{V}(U) = C^k(\overline{U}, \mathbf{C}^m)$.

Example 2.1.2. For the Hölder norm $C^{k,\gamma}$, $k \geq 0$ and $0 < \gamma < 1$, $\mathcal{V}(U) = C^{k,\alpha}(\overline{U}, \mathbf{C}^m)$.

Example 2.1.3. For the L^p norm $||\cdot||_{L^p}$, $1 \leq p < \infty$, $\mathcal{V}(U) = L^p(U, \mathbf{C}^m)$.

Example 2.1.4. For the Sobolev norm $||\cdot||_{W^{k,p}}$, $k \geq 0$ and $1 \leq p < \infty$, $\mathcal{V}(U) = W^{k,p}(U, \mathbf{C}^m)$.

Now suppose E is a rank r $\mathbf{C}$-vector bundle over M and let π be its projection map. Let $||\cdot||$ be a norm as above. There is a standard way to extend the norm $||\cdot||$ to certain local sections of E. First pick a finite atlas $\{(V_i, \varphi_i)\}$ of M with the property that for each index i there is a C^∞ function $\psi_i : \pi^{-1}(V_i) \to \mathbf{C}^r$ such that $\pi \times \psi_i$ is a trivialization of E over V_i. Next, let $\{\rho_i\}$ be a smooth partition of unity subordinate to $\{V_i\}$. A (rough) section ω is *admissible* if $\psi_i \circ (\rho_i \cdot \omega) \circ \varphi_i{}^{-1} \in \mathcal{V}(\varphi_i(V_i))$ for each index i. The space of all admissible sections is denoted by $F(M, E, ||\cdot||)$. For $\omega \in F(M, E, ||\cdot||)$, we define

$$||\omega|| = \sum_i ||\psi_i \circ (\rho_i \cdot \omega) \circ \varphi_i{}^{-1}||.$$

Clearly, the space $F(M, E, ||\cdot||)$ is Banach under the norm $||\cdot||$. Note that some choices have been made in defining $F(M, E, ||\cdot||)$ and $||\cdot||$. However, it can be shown that neither $F(M, E, ||\cdot||)$ nor the equivalence class of $||\cdot||$ depend on those choices. We will therefore find it justified to not specify those choices when mentionning such a norm.

Example 2.1.5. For the C^k norm $||\cdot||_{C^k}$, $k \geq 0$, we have $F(M, E, ||\cdot||_{C^k}) = C^k(M, E)$.

Example 2.1.6. For the Hölder norm $C^{k,\gamma}$, $k \geq 0$ and $0 < \gamma < 1$, we write $C^{k,\gamma}(M, E)$ instead of $F(M, E, ||\cdot||_{C^{k,\gamma}})$.

Example 2.1.7. For the L^p norm $||\cdot||_{L^p}$, $1 \leq p < \infty$, we write $L^p(M, E)$ instead of $F(M, E, ||\cdot||_{L^p})$.

Example 2.1.8. For the Sobolev norm $||\cdot||_{W^{k,p}}$, $k \geq 0$ and $1 \leq p < \infty$, we write $W^{k,p}(M, E)$ instead of $F(M, E, ||\cdot||_{W^{k,p}})$. As is customary, when there is no risk of confusion with another symbol, we write H^k instead of $W^{k,2}$.

The Sobolev embedding theorem has the following implication: if $2r > n$, then $W^{k+r,2}(M, E) \subseteq C^k(M, E)$ and this inclusion is continuous with respect to the norms $||\cdot||_{H^{k+r}}$ and $||\cdot||_{C^k}$. Thus $C^\infty(M, E) = \bigcap_{k\geq 0} W^{k,2}(M, E)$.

Note that in the examples above the spaces of sections given are Banach when paired with their corresponding norms. Furthermore, $C^\infty(M, E)$ is contained and dense in all those spaces.

Let h be a Hermitian metric on E, i.e. a "smoothly varying Hermitian inner product on the fibers of E." Suppose M is oriented and let g be a Riemannian metric on M. Let dV be the corresponding volume form. We define the inner product $\langle \alpha, \beta \rangle_h$ of two sections α and β of E as

$$\langle \alpha, \beta \rangle_h = \int_M h(\alpha, \beta)\, dV = \int_M \Re(h(\alpha, \beta))\, dV + \sqrt{-1} \int_M \Im(h(\alpha, \beta))\, dV,$$

whenever this quantity exists. Let $||\cdot||_h$ be the norm corresponding to this product. It may be fairly easily shown that the space of sections on which $||\cdot||_h$ is defined is exactly $L^2(M, E)$ and that in fact $||\cdot||_h$ is equivalent to the norm $||\cdot||_{L^2}$.

2.2 Linear differential operators

In this section we introduce linear differential operators on sections of vector bundles. As we will see, these operators are locally linear differential operators in the classical sense. For $U \subseteq \mathbf{R}^n$ open, a *linear differential operator over U* is an operator of the form

$$L = \sum_{|\alpha| \leq s} a_\alpha(x) \cdot D^\alpha,$$

where for each α, a_α is a $k \times l$ matrix whose entries are $\mathbf{C}$-valued functions defined in U. Thus L takes sufficiently differentiable $\mathbf{C}^l$-valued functions over U to the space of $\mathbf{C}^k$-valued functions over U. The *order* of L, denoted by ord L, is the smallest integer s which can appear in the above expression for L; by convention the order of the 0 operator is $-\infty$.

For the rest of this section, M is oriented, g is a Riemannian metric on M, E and F are $\mathbf{C}$-vector bundles over M, and h_1 and h_2 are Hermitian metrics on E and F respectively.

We denote the volume form corresponding to the orientation on M and the metric g by dV.

A *linear differential operator (between sections of E and sections of F)* is a $\mathbf{C}$-linear mapping

$$P : C^{\infty}(M, E) \to C^{\infty}(M, F)$$

with the property that for any section $\omega \in C^{\infty}(M, E)$, one has

$$\operatorname{supp} P\omega \subseteq \operatorname{supp} \omega.$$

There is an important result concerning the local behaviour of such an operator due to Peetre.

Theorem 2.2.1 (Theorem of Peetre). *Around each point $y \in M$ there is an open neighbourhood $U \subseteq M$ diffeomorphic to an open subset V of $\mathbf{R}^n$ and over which E and F are trivial such that the mapping that P induces from $C^{\infty}(V, \mathbf{C}^{\operatorname{rank} E})$ to $C^{\infty}(V, \mathbf{C}^{\operatorname{rank} F})$ is a linear differential operator in the classical sense, i.e. an operator of the form*

$$\sum_{|\alpha| \leq s} a_{\alpha}(x) \cdot D^{\alpha},$$

where for each α, a_{α} is a $\operatorname{rank} F \times \operatorname{rank} E$ *matrix whose entries are functions in* $C^{\infty}(V, \mathbf{C})$. ∎

Thus, as stated before, a linear differential operator as we have just defined is locally a linear differential operator in the classical sense. The order of a localization of P around a point $y \in M$ such as in the statement of the above theorem is an invariant, called the *order of P at y*. Since M is compact, there is a maximal such order, which is simply called the *order* of P and is denoted by $\operatorname{ord} P$. Note that $\operatorname{ord} 0 = -\infty$.

Example 2.2.1. Suppose n is even and that J is an integrable ACS on M so that (M, J)

is a complex manifold. Let E be a holomorphic vector bundle over M. The operators

$$d : C^\infty(M, \bigwedge^r T^{*\mathbf{C}}M) \to C^\infty(M, \bigwedge^{r+1} T^{*\mathbf{C}}M),$$
$$\partial : C^\infty(M, \bigwedge^{p,q} T^{*\mathbf{C}}M) \to C^\infty(M, \bigwedge^{p+1,q} T^{*\mathbf{C}}M),$$
$$\overline{\partial} : C^\infty(M, \bigwedge^{p,q} T^{*\mathbf{C}}M) \to C^\infty(M, \bigwedge^{p,q+1} T^{*\mathbf{C}}M),$$
$$\overline{\partial} : \mathcal{A}^{p,q}E \to \mathcal{A}^{p,q+1}E,$$

are linear differential operators which are either the 0 operator or of order 1.

An important consequence of Theorem 2.2.1 is that P extends unambiguously to linear operators

$$P : C^{k+\operatorname{ord} P}(M, E) \to C^k(M, F),$$
$$P : C^{k+\operatorname{ord} P,\gamma}(M, E) \to C^{k,\gamma}(M, F),$$
$$P : W^{k+\operatorname{ord} P,p}(M, E) \to W^{k,p}(M, F),$$

for every $k \geq 0$, $1 \leq p < \infty$ and $0 < \gamma < 1$. The last extension above is made possible by interpreting differentiation in the weak sense. Since M is compact, we have, for $\varphi \in C^{k+\operatorname{ord} P}(M, E)$,

$$||P\varphi||_{C^k} \leq C_1||\varphi||_{C^{k+\operatorname{ord} P}},$$
$$||P\varphi||_{C^{k,\gamma}} \leq C_2||\varphi||_{C^{k+\operatorname{ord} P,\gamma}},$$
$$||P\varphi||_{W^{k,p}} \leq C_3||\varphi||_{W^{k+\operatorname{ord} P,p}},$$

where $C_1, C_2, C_3 > 0$ are constants independent of φ.

A linear differential operator $Q : C^\infty(M, F) \to C^\infty(M, E)$ is said to be a formal adjoint of P if for every $\omega \in C^\infty(M, E)$ and $\varphi \in C^\infty(M, F)$, one has

$$\langle P\omega, \varphi\rangle_{h_2} = \langle \omega, Q\varphi\rangle_{h_1}.$$

It can be shown that such a Q always exists: one invokes Theorem 2.2.1 and then uses a partition of unity and integration by parts. Note that Q is necessarily unique.

Thus there is one and only one formal adjoint of P and we denote it by P^*. Note that $(P^*)^* = P$. If $E = F$, $h_1 = h_2$ and $P^* = P$, we say that P is *self-adjoint*. The following proposition is easily proven:

Proposition 2.2.1. *The following statements hold:*

(i) $\operatorname{ord} PQ \leq \operatorname{ord} P + \operatorname{ord} Q$,

(ii) $(PQ)^* = Q^* P^*$,

(iii) $\operatorname{ord} P^* = \operatorname{ord} P$.

∎

2.2.1 Principal symbols and elliptic operators

An important kind of linear differential operators are those called *elliptic*. To define them we first need to define principal symbols.

Let $U \subseteq \mathbf{R}^n$ be open and let

$$L = \sum_{|\alpha| \leq s} a_\alpha(x) \cdot D^\alpha$$

be a linear differential operator over U, where each a_α is an $n \times n$ matrix. Fix $x \in U$. Let $\xi \in \mathbf{R}^n \backslash \{0\}$. The *principal symbol* $\sigma_\xi(L; x)$ is the matrix

$$(\sqrt{-1})^s \sum_{|\alpha| = s} a_\alpha(x)\, \xi^\alpha. \tag{2.2.1}$$

We say that L is *elliptic at* $x \in U$ if the principal symbol $\sigma_\xi(L; x)$ is an isomorphism for every $\xi \in \mathbf{R}^n \backslash \{0\}$. The operator L is said to be *elliptic in* U if it is elliptic at every $x \in U$.

Suppose again that E and F are $\mathbf{C}$-vector bundles over M, this time with rank E = rank F. Let $P : C^\infty(M, E) \to C^\infty(M, F)$ be a linear differential operator of order s. Then by using Theorem 2.2.1 we can make sense of the principal symbol $\sigma_\xi(P; x)$ for $x \in M$ and $\xi \in \mathbf{R}^n \backslash \{0\}$. However, this interpretation depends on various choices. This said, there is an invariant definition of the principal symbol $\sigma_\xi(P; x)$, with $\xi \in T_x^* M$ instead, which coincides with this interpretation after identifying $T_x^* M$ with $\mathbf{R}^n$ via the diffeomorphism mentioned in the statement of Theorem 2.2.1. Let $u \in E_x$ and $\xi \in T_x^* M$. Choose $\omega \in C^\infty(M, E)$ such that $\omega(x) = u$ and $f \in C^\infty(M)$ such that $f(x) = 0$ and $df(x) = \xi$. The *principal symbol* $\sigma_\xi(P; x)$ is the $\mathbf{C}$-linear map from E_x to F_x defined by

$$\sigma_\xi(P; x)\, u = \frac{(\sqrt{-1})^s}{s!} \, P(f^s \omega)|_x .$$

It may be shown that this definition does not depend on ω or f. Ellipticity is naturally defined as follows: P is said to be elliptic at $x \in M$ if $\sigma_\xi(P; x)$ is invertible for every $\xi \in T_x^* M \backslash \{0\}$. We say that P is *elliptic* if it is elliptic at every $x \in M$. When P is identified, around a point, with a linear differential operator in the classical sense as in Theorem 2.2.1 the two notions of ellipticity given coincide.

Let G be another $\mathbf{C}$-vector bundle over M, and $Q : C^\infty(M, E) \to C^\infty(M, F)$ and $R : C^\infty(M, F) \to C^\infty(M, G)$ linear differential operators. We have:

(i) $\sigma_\xi(P + Q; x) = \sigma_\xi(P; x) + \sigma_\xi(Q; x)$,

(ii) $\sigma_\xi(RP; x) = \sigma_\xi(R; x) \circ \sigma_\xi(P; x)$.

(iii) $\sigma_\xi(P^*; x) = \sigma_\xi(P; x)^*$ (the Hermitian adjoint of $\sigma_\xi(P; x)$).

The following proposition will be of use to us:

Proposition 2.2.2. *If* $P : C^\infty(M, E) \to C^\infty(M, F)$ *and* $Q : C^\infty(M, F) \to$

$C^\infty(M,G)$ are of order 1 then the linear differential operator of order 2

$$PP^* + Q^*Q : C^\infty(M,F) \to C^\infty(M,F)$$

is elliptic at $x \in M$ if the following sequence is exact for every $\xi \in T_x^*M\backslash\{0\}$:

$$E_x \overset{\sigma_\xi(P;x)}{\longrightarrow} F_x \overset{\sigma_\xi(Q;x)}{\longrightarrow} G_x.$$

■

2.2.2 Schauder and L^p estimates

Let E and F be $\mathbb{C}$-vector bundles over M. Suppose $P : C^\infty(M,E) \to C^\infty(M,F)$ is an elliptic operator of order s.

Theorem 2.2.2. *There are constants $C_1, C_2, \ldots, C_6$ such that, for every $C^{s+k,\gamma}(M,E)$ and every $\varphi \in W^{s+k,p}(M,E)$,*

(a) *(Schauder estimates)*

$$||\omega||_{C^{s+k,\gamma}} \le C_1||P\omega||_{C^{k,\gamma}} + C_2||\omega||_{C^0} \le C_3||\omega||_{C^{s+k,\gamma}},$$

(b) *(L^p estimates)*

$$||\varphi||_{W^{s+k,p}} \le C_4||P\varphi||_{W^{k,p}} + C_5||\varphi||_{L^1} \le C_6||\varphi||_{W^{s+k,p}}.$$

Moreover, if $p = 2$ and ω and φ are chosen so that they are orthogonal to $\ker P$ *with respect to $\langle\cdot,\cdot\rangle_{L^2}$, then we can let $C_2 = C_5 = 0$.* ■

One consequence of the above theorem is the following crucial result:

Corollary 2.2.1. *Consider the extensions of P above. Then in both cases,* $\ker P \subseteq C^\infty(M,E)$. *Furthermore, both* $\ker P$ *and* $\ker P^* \subseteq C^\infty(M,F)$ *are finite dimensional. Finally, we have the following orthogonal decompositions for $k \ge 0$, $0 < \gamma < 1$ and $1 < p < \infty$:*

(a)

$$W^{k,p}(M,F) = P(W^{s+k,p}(M,E)) \oplus \ker P^*,$$

(b)

$$C^{k,\gamma}(M,F) = P(C^{s+k,\gamma}(M,E)) \oplus \ker P^*,$$

(c)

$$C^{\infty}(M,F) = P(C^{\infty}(M,E)) \oplus \ker P^*.$$

■

2.2.3 Elliptic regularity

Let $U \subseteq \mathbf{R}^n$ be open and consider a differential operator of the form

$$L = \sum_{|\alpha| \leq s} a_\alpha(x) \cdot D^\alpha$$

where for each α, a_α is a $k \times l$ matrix of $\mathbf{C}$-valued functions defined in U, with at least one a_α with $\alpha = s$ being not identically 0. Suppose L is elliptic in U. The following result says that a solution $u \in W^{s,p}(U, \mathbf{C}^l)$, with $1 < p < \infty$, to the equation $Lu = f$, where f is a $\mathbf{C}^k$-valued function defined in U, is "as regular as f and the coefficients of L allow it to be."

Theorem 2.2.3 (Regularity). *Let $k \geq 0$, $0 < \gamma < 1$ and $1 < p < \infty$. Let L, U and f be as above. Suppose $u \in W^{s,p}(U, \mathbf{C}^l)$ satisfies $Lu = f$ almost everywhere. Then regularity conditions on a_α, $|\alpha| \leq s$, and f entail regularity conditions on u in accordance to the following table:*

a_α, $\lvert\alpha\rvert \le s$	f	u
C^k	$W^{k,p}$, $1 < p < \infty$	$W^{k+s,p}$
$C^{k,\gamma}$	$C^{k,\gamma}$	$C^{k+s,\gamma}$
C^∞	C^∞	C^∞
C^ω	C^ω	C^ω

■

This theorem has particularly pleasant consequences when applied to certain non-linear equations. Suppose for example that $u \in C^2(U, \mathbf{C})$, $U \subseteq \mathbf{R}^n$ open, solves

$$\sum_{|\alpha|\le 2} a_\alpha(x, D^l u(x)) D^\alpha u = 0$$

in U, where $|l| = 1$ and $a_\alpha \in C^\infty(U \times \mathbf{C}, \mathbf{C})$ for each α. Suppose furthermore that the operator

$$\sum_{|\alpha|\le 2} a_\alpha(x, D^l u(x)) D^\alpha$$

is of order 2 and elliptic. Clearly, the functions $a_\alpha(x, D^l u(x))$ are C^1, and thus $C^{0,\gamma}$ for $0 < \gamma < 1$. Using Theorem 2.2.3, we see that u is in fact $C^{2,\gamma}$. In turn, this means that the functions $a_\alpha(x, D^l u(x))$ are $C^{1,\gamma}$. Using Theorem 2.2.3 again, we see that u is $C^{3,\gamma}$. Continuing this way, we discover that u is actually C^∞. This kind of argument, by which one deduces higher regularity than is initially known for a solution of a non-linear partial differential equation such as above, is known as "bootstrapping."

2.3 Hodge theory

One important application of the theory of elliptic linear partial differential operators is Hodge theory. Suppose J is an integrable ACS on M so that (M, J) is a complex manifold and E is a holomorphic vector bundle over M. Suppose we have a family of Hermitian metrics $\{h_{p,q}\}$ corresponding to the bundles $\bigwedge^{p,q} T^{*\mathbf{C}} M \otimes_{\mathbf{C}} E$. As there will be no ambiguity, we suppress the subscripts and simply write h for anyone of the metrics $h_{p,q}$. Let g a Riemannian metric on M.[1] Along with the natural orientation on M, this gives us an inner product $\langle\cdot,\cdot\rangle_h$ on the sections of $\bigwedge^{p,q} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} E$.

Let $x \in M$ and $\xi \in T_x^*M\backslash\{0\}$. Then in view of the Leibniz rule (1.2.3)

$$\sigma_\xi(\overline{\partial}; x)\, u = \sqrt{-1}\xi^{0,1} \wedge u$$

for $u \in \bigwedge^{p,q} T_x^{*\mathbf{C}}M \otimes_{\mathbf{C}} E_x$, where $\xi^{0,1}$ is the projection of ξ onto $\bigwedge^{0,1} T_x^{*\mathbf{C}}M$. From this it follows that the sequence

$$\bigwedge^{p,q-1} T_x^{*\mathbf{C}}M \otimes_{\mathbf{C}} E_x \overset{\sigma_\xi(\overline{\partial};x)}{\longrightarrow} \bigwedge^{p,q} T_x^{*\mathbf{C}}M \otimes_{\mathbf{C}} E_x \overset{\sigma_\xi(\overline{\partial};x)}{\longrightarrow} \bigwedge^{p,q+1} T_x^{*\mathbf{C}}M \otimes_{\mathbf{C}} E_x$$

is exact. Making use of Proposition 2.2.2, the operator

$$\Delta^{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial},$$

which takes $\mathcal{A}^{p,q}E$ into itself, is either the 0 operator or elliptic of order 2. This operator is called the *Laplace operator*. Obviously, it is self-adjoint. We say that $\varphi \in \mathcal{A}^{p,q}E$ is *h-harmonic* (or simply *harmonic* if h is clear from the context) if $\Delta^{\overline{\partial}}\varphi = 0$. Since

$$\left\langle \Delta^{\overline{\partial}}\varphi, \varphi \right\rangle_h = \left\langle \overline{\partial}\,\overline{\partial}^*\varphi, \varphi \right\rangle_h + \left\langle \overline{\partial}^*\,\overline{\partial}\varphi, \varphi \right\rangle_h = \left\langle \overline{\partial}^*\varphi, \overline{\partial}^*\varphi \right\rangle_h + \left\langle \overline{\partial}\varphi, \overline{\partial}\varphi \right\rangle_h,$$

φ is harmonic if and only if $\overline{\partial}\varphi = 0$ and $\overline{\partial}^*\varphi = 0$. We denote the space of harmonic (p,q)-forms by $\mathcal{H}_h^{p,q}(E)$. Corollary 2.2.1 gives

Theorem 2.3.1 (Hodge decomposition theorem). *The space $\mathcal{H}_h^{p,q}$ is finite-dimensional. Furthermore, we have the orthogonal decomposition*

$$\mathcal{A}^{p,q}E = \Delta^{\overline{\partial}}\mathcal{A}^{p,q}E \oplus \mathcal{H}_h^{p,q}(E).$$

[1] Usually, one starts with Hermitian metrics h_1 and h_2 on $T^{1,0}M$ and E respectively. Then h_1 is used to naturally define a Hermitian metric on $\bigwedge^{p,q} T^{*\mathbf{C}}M$, also denoted by h_1. The metric h is then defined as $h_1 \otimes h_2$ and g is defined as the Riemannian metric on M corresponding to h_1 via the $\mathbf{R}$-vector bundle isomorphism between TM and $T^{1,0}M$ given by $X \mapsto X - \sqrt{-1}\, J|_x(X)$, for $x \in M$ and $X \in T_xM$.

■

Corollary 2.3.1. *We have the orthogonal decomposition*

$$\mathcal{A}^{p,q}E = \overline{\partial}\,\mathcal{A}^{p,q-1}E \oplus \overline{\partial}^*\,\mathcal{A}^{p,q+1}E \oplus \mathcal{H}_h^{p,q}(E).$$

Proof. Let $\psi \in \mathcal{A}^{p,q-1}E$ and $\omega \in \mathcal{A}^{p,q+1}E$. We have

$$\left\langle \overline{\partial}\,\psi, \overline{\partial}^*\,\omega \right\rangle_h = \left\langle \overline{\partial}^2\,\psi, \omega \right\rangle_h = 0.$$

Thus $\overline{\partial}\,\mathcal{A}^{p,q-1}E \perp \overline{\partial}^*\,\mathcal{A}^{p,q+1}E$. If $\varphi \in \mathcal{A}^{p,q}E$ is harmonic, then

$$\left\langle \varphi, \overline{\partial}\,\psi \right\rangle_h = \left\langle \overline{\partial}^*\,\varphi, \psi \right\rangle_h = 0.$$

So $\mathcal{H}_h^{p,q}(E) \perp \overline{\partial}\,\mathcal{A}^{p,q-1}E$. Likewise, $\mathcal{H}_h^{p,q}(E) \perp \overline{\partial}^*\,\mathcal{A}^{p,q+1}E$. So the spaces $\overline{\partial}\,\mathcal{A}^{p,q-1}E$, $\overline{\partial}^*\,\mathcal{A}^{p,q+1}E$ and $\mathcal{H}_h^{p,q}(E)$ are mutually orthogonal. Since

$$\Delta^{\overline{\partial}}\,\mathcal{A}^{p,q}E \subseteq \overline{\partial}\,\mathcal{A}^{p,q-1}E \oplus \overline{\partial}^*\,\mathcal{A}^{p,q+1}E,$$

the statement follows. ■

Let $\varphi \in \mathcal{A}^{p,q}E$. By the above, $\varphi = \overline{\partial}\,\psi + \overline{\partial}^*\,\omega + \eta$, where $\psi \in \mathcal{A}^{p,q-1}E$, $\omega \in \mathcal{A}^{p,q+1}E$ and $\eta \in \mathcal{H}_h^{p,q}(E)$. Suppose $\varphi \in \ker \overline{\partial}_{p,q}$. Note that $\overline{\partial}\,\varphi = \overline{\partial}\,\overline{\partial}^*\,\omega = 0$. Since

$$\left\langle \overline{\partial}^*\,\omega, \overline{\partial}^*\,\omega \right\rangle_h = \left\langle \overline{\partial}\,\overline{\partial}^*\,\omega, \omega \right\rangle_h = 0,$$

we have $\overline{\partial}^*\,\omega = 0$. Conversely if $\overline{\partial}^*\,\omega = 0$, then $\varphi \in \ker \overline{\partial}_{p,q}$. Thus

$$\ker \overline{\partial}_{p,q} = \overline{\partial}\,\mathcal{A}^{p,q-1}E \oplus \mathcal{H}_h^{p,q}(E).$$

So

$$\mathcal{H}_h^{p,q}(E) \simeq H_{\overline{\partial}}^{p,q}(M, E).$$

If $\Theta^p E$ denotes the sheaf of holomorphic sections of $\bigwedge^{p,q} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} E$, by Theorem 1.3.2 we obtain

Theorem 2.3.2 (Hodge-Dolbeaut theorem). *We have*

$$\mathcal{H}_h^{p,q}(E) \simeq H^q(M, \Theta^p E), \quad q \geq 1.$$

■

Given a class in $H^q(M, \Theta^p E)$, the element in $\mathcal{H}_h^{p,q}(E)$ to which it corresponds via the preceding isomorphism is called its *harmonic representative*.

Fix p and q. The projection of $\mathcal{A}^{p,q}E$ onto $\mathcal{H}_h^{p,q}(E)$ along $\Delta^{\bar\partial}\mathcal{A}^{p,q}E$ is denoted by H. Given $\varphi \in \mathcal{A}^{p,q}E$, we have

$$\varphi = H\varphi + \Delta^{\bar\partial}\psi \tag{2.3.1}$$

for some $\psi \in \mathcal{A}^{p,q}E$. In turn, $\psi = H\psi + \Delta^{\bar\partial}\psi_1$ for some $\psi_1 \in \mathcal{A}^{p,q}E$. Now $\Delta^{\bar\partial}\psi = (\Delta^{\bar\partial})^2\psi_1$. We may thus replace ψ by $\Delta^{\bar\partial}\psi_1$ in (2.3.1) and assume that $\psi \perp \mathcal{H}_h^{p,q}(E)$. In fact, such a ψ is unique. To see this, suppose ψ' is another contender. Then $\Delta^{\bar\partial}(\psi - \psi') = 0$, so $\psi - \psi' \in \mathcal{H}_h^{p,q}(E)$. But $\psi - \psi' \perp \mathcal{H}_h^{p,q}(E)$, and so $\psi - \psi' = 0$, i.e. $\psi = \psi'$. The linear operator which sends φ to the unique such ψ is called the *Green's operator* and is denoted by G. Thus

$$\varphi = H\varphi + \Delta^{\bar\partial}G\varphi, \tag{2.3.2}$$

which is known as *Poisson's equation.*

Proposition 2.3.1. *The operator G commutes with any operator that commutes with $\Delta^{\bar\partial}$.*

Proof. Suppose $T : \mathcal{A}^{p,q}E \to \mathcal{A}^{r,s}E$ is an operator with $\Delta^{\bar\partial}T = T\Delta^{\bar\partial}$. Let $\varphi \in \mathcal{A}^{p,q}E$. Applying T to (2.3.2), we find

$$T\varphi = TH\varphi + T\Delta^{\bar\partial}G\varphi = TH\varphi + \Delta^{\bar\partial}TG\varphi. \tag{2.3.3}$$

At the same time, by (2.3.2) again, we have

$$T\varphi = HT\varphi + \Delta^{\bar\partial}GT\varphi. \tag{2.3.4}$$

But $\Delta^{\bar{\partial}} TH\varphi = T\,\Delta^{\bar{\partial}} H\varphi = 0$. Thus $TH\varphi \in \mathcal{H}_h^{r,s}(E)$. Since $\mathcal{H}_h^{r,s}(E) \perp \Delta^{\bar{\partial}} \mathcal{A}^{r,s}E$, (2.3.3) and (2.3.4) give us $TH\varphi = HT\varphi$ and $\Delta^{\bar{\partial}} TG\varphi = \Delta^{\bar{\partial}} GT\varphi$. Thus

$$T\varphi = HT\varphi + \Delta^{\bar{\partial}} TG\varphi.$$

By definition then, $GT\varphi = TG\varphi$. ■

Proposition 2.3.2. *The following identities hold:*

$$\bar{\partial} H = H\,\bar{\partial} = 0,$$
$$\bar{\partial}^* H = H\,\bar{\partial}^* = 0,$$
$$GH = HG = 0,$$
$$\bar{\partial} G = G\,\bar{\partial},$$
$$\bar{\partial}^* G = G\,\bar{\partial}^*,$$
$$\Delta^{\bar{\partial}} G = G\,\Delta^{\bar{\partial}}.$$

Proof. Only the first three identities must be verified, as the others are immediate corollaries of the previous proposition. Let $\varphi \in \mathcal{A}^{p,q}E$. One has $H\varphi \in \{\psi : \bar{\partial}\psi = \bar{\partial}^*\psi = 0\}$ and so $\bar{\partial} H\varphi = 0$. On the other hand since $\Delta^{\bar{\partial}} H\,\bar{\partial}\varphi = 0$, one has $GH\,\bar{\partial}\varphi = 0$. This then means $H\,\bar{\partial} G\varphi = 0$. Since φ is arbitrary, this means that $H\,\bar{\partial} = 0$. The second identity is proved in a similar manner. Finally, since $\Delta^{\bar{\partial}} H = H\,\Delta^{\bar{\partial}} = 0$ one has $GH = HG$. By definition, it is clear that $GH = 0$. This proves the third identity. ■

CHAPTER III

THE SPACE OF ALMOST COMPLEX STRUCTURES

In this chapter, we describe the space of almost complex structures, first on a vector space, and then on a C^∞ manifold. This is an essential step in the development of our framework for deformation theory.

3.1 On a vector space

Let V be a real vector space of dimension $2n$. We wish to examine the space $\mathcal{J}(V) \subseteq \mathrm{End}(V)$ of almost complex structures on V. Suppose $J, J' \in \mathcal{J}(V)$ and let $B = \{e_1, \ldots, e_n, J(e_1), \ldots, J(e_n)\}$ and $B' = \{e'_1, \ldots, e'_n, J'(e'_1), \ldots, J'(e'_n)\}$ be bases of V as in Proposition 1.1.2. Let S be the linear map defined by $e_i \mapsto e'_i$ and $J(e_i) \mapsto J'(e'_i)$. Then $J' = SJS^{-1}$. Conversely, if S is any transformation in $\mathrm{GL}(V)$ and $J \in \mathcal{J}(V)$, then $SJS^{-1} \in \mathcal{J}(V)$. In other words, $\mathcal{J}(V)$ constitutes an orbit of the action of $\mathrm{GL}(V)$ on $\mathrm{End}(V)$ by conjugation. There is therefore a unique C^∞ manifold structure on $\mathcal{J}(V)$ turning it into a homogeneous $\mathrm{GL}(V)$-space under this action.

The above being said, we will find it preferable to explicitly construct an atlas for $\mathcal{J}(V)$, using the so-called Cayley correspondence. The reason for this is that we will want to mimic this construction later, when we will be working with Fréchet manifolds and may thus not appeal to standard finite-dimensional differential geometry.

3.1.1 The Cayley correspondence

Fix $J_0 \in \mathcal{J}(V)$. If J is another ACS, to distinguish among the eigenspaces of J_0 and J we will make use of subscripts so that e.g. $V^{1,0}_{J_0}$ is the $\sqrt{-1}$-eigenspace of J_0. An ACS J on V is said to be *commensurable* to J_0 if the projection of $V^{0,1}_J$ onto $V^{0,1}_{J_0}$ along $V^{1,0}_{J_0}$ is an isomorphism. Equivalently, J is commensurable to J_0 if $V^{0,1}_J \cap V^{1,0}_{J_0}$ is trivial. Suppose J is commensurable to J_0 and let $\mathrm{pr}^{1,0}$ and $\mathrm{pr}^{0,1}$ be the projections of $V^{\mathbf{C}}$ onto $V^{1,0}_{J_0}$ and $V^{0,1}_{J_0}$ respectively. By definition, $\mathrm{pr}^{0,1}$ restricts to a isomorphism $\mathrm{pr}^{0,1} : V^{0,1}_J \to V^{0,1}_{J_0}$. Define $\nu : V^{0,1}_{J_0} \to V^{1,0}_{J_0}$ as $\nu = \mathrm{pr}^{1,0} \circ (\mathrm{pr}^{0,1})^{-1}$. It is easy to see that

$$V^{0,1}_J = \{Z + \nu(Z) \ : \ Z \in V^{0,1}_{J_0}\}. \tag{3.1.1}$$

We know that $\mathrm{pr}^{1,0}$ restricts to $\mathbf{C}$-linear isomorphism of (V, J_0) into $V^{1,0}_{J_0}$ and $\mathrm{pr}^{0,1}$ to a $\mathbf{C}$-linear isomorphism of $(V, -J_0)$ into $V^{0,1}_{J_0}$. The composition $(\mathrm{pr}^{1,0}|_V)^{-1}\nu\ \mathrm{pr}^{0,1}$ gives an endomorphism of V, which we denote by μ. For $X \in V$, one has

$$\begin{aligned}
\mu J_0(X) &= (\mathrm{pr}^{1,0}|_V)^{-1}\nu\,\mathrm{pr}^{0,1}\ J_0(X)\\
&= (\mathrm{pr}^{1,0}|_V)^{-1}\nu(-\sqrt{-1}\ \mathrm{pr}^{0,1}(X))\\
&= (\mathrm{pr}^{1,0}|_V)^{-1}(-\sqrt{-1}\nu\ \mathrm{pr}^{0,1}(X))\\
&= -J_0\mu(X).
\end{aligned}$$

Thus μ is J_0-antilinear. For an endomorphism $\eta : V \to V$, the space of η-antilinear endomorphisms of V is denoted by $\mathrm{Ant}_\eta(V)$, so that $\mu \in \mathrm{Ant}_{J_0}(V)$.

Conversely, let μ be a J_0-antilinear endomorphism of V. Let $\nu = \mathrm{pr}^{1,0}\ \mu\ (\mathrm{pr}^{0,1}|_V)^{-1}$; this is a $\mathbf{C}$-linear morphism of $V^{0,1}_{J_0}$ into $V^{1,0}_{J_0}$. Let D_μ be the subspace of $V^{\mathbf{C}}$ defined by the RHS of (3.1.1). There is at most one ACS J such that $D_\mu = V^{0,1}_J$ and it exists if and only if $D_\mu \cap \overline{D_\mu} = \{0\}$. When J exists, $V^{0,1}_J \cap V^{1,0}_{J_0} = \{0\}$ and so J is commensurable to J_0.

Proposition 3.1.1. *The condition $D_\mu \cap \overline{D_\mu} = \{0\}$ is equivalent to the invertibility of*

$\mathrm{Id}-\mu$.

Proof. Suppose first $\mathrm{Id}-\mu$ is not invertible. Then there is a nonzero $X \in V$ with $\mu(X) = X$ and it is easy to see that $X \in D_\mu \cap \overline{D_\mu}$. Conversely, suppose $D_\mu \cap \overline{D_\mu} \neq \{0\}$. Since

$$\overline{D_\mu} = \{Z + \overline{\nu(\overline{Z})} \;:\; Z \in V_{J_0}^{1,0}\},$$

this means there are $Z \in V_{J_0}^{0,1}$ and $Z' \in V_{J_0}^{1,0}$, both nonzero, such that

$$Z + \nu(Z) = Z' + \overline{\nu(\overline{Z'})},$$

from which we see that $Z' = \nu(\overline{\nu(\overline{Z'})}) = \mathrm{pr}^{1,0}\,\mu^2(\mathrm{pr}^{1,0}|_V)^{-1}(Z')$. Thus

$$\mathrm{pr}^{1,0}(\mathrm{Id}-\mu^2)(\mathrm{pr}^{1,0}|_V)^{-1}(Z') = 0.$$

In other words, $\mathrm{Id}-\mu^2$ is not invertible. But $\mathrm{Id}-\mu^2 = (\mathrm{Id}+\mu)(\mathrm{Id}-\mu)$ and $\mathrm{Id}+\mu = -J_0(\mathrm{Id}-\mu)J_0$, and so $\mathrm{Id}-\mu$ is not invertible. ■

Remark 3.1.1. As can be gleaned from the proof above, for any $J \in \mathcal{J}(M)$ and $\mu \in \mathrm{Ant}_J(V)$, the invertibility of one of $\mathrm{Id}-\mu$, $\mathrm{Id}+\mu$ or $\mathrm{Id}-\mu^2$ implies that of the other two.

The above effectively shows that to an ACS J commensurable to J_0 we can associate an endomorphism μ such that $\mathrm{Id}+\mu$ is invertible and vice-versa. These associations are obviously inverse to each other and thus there is a one-to-one correspondence

$$\left\{ \text{ ACS's commensurable to } J_0. \right\} \longleftrightarrow \left\{ \begin{array}{l} \mu \in \mathrm{Ant}_{J_0}(V) \text{ such that} \\ \mathrm{Id}+\mu \text{ is invertible.} \end{array} \right\}, \qquad (3.1.2)$$

called the *Cayley correspondence*. Let J be an ACS commensurable to J_0 and μ its corresponding J_0-antilinear endomorphism of V. Let $\nu = \mathrm{pr}^{1,0}\,\mu\,(\mathrm{pr}^{0,1}|_V)^{-1}$. Now let $X \in V$ and $\tilde{X} = (\mathrm{Id}+\mu)^{-1}(X)$. We write $\tilde{X} = \tilde{X}^{1,0} + \tilde{X}^{0,1}$ where $\tilde{X}^{1,0} = \mathrm{pr}^{1,0}(\tilde{X})$ and $\tilde{X}^{0,1} = \mathrm{pr}^{0,1}(\tilde{X})$. We have

$$\mu(\tilde{X}) = \mu(\tilde{X}^{1,0} + \tilde{X}^{0,1}) = \nu(\tilde{X}^{0,1}) + \overline{\nu(\tilde{X}^{0,1})}.$$

Thus

$$X = (\mathrm{Id} + \mu)(\tilde{X}) = \tilde{X}^{0,1} + \nu(\tilde{X}^{0,1}) + \overline{\tilde{X}^{0,1} + \nu(\tilde{X}^{0,1})}.$$

This means that $\tilde{X}^{0,1} + \nu(\tilde{X}^{0,1})$ is the projection of X onto $V_J^{0,1}$. We thus have

$$\begin{aligned} J(X) &= -\sqrt{-1}(\tilde{X}^{0,1} + \nu(\tilde{X}^{0,1})) + \sqrt{-1}\,\overline{\tilde{X}^{0,1} + \nu(\tilde{X}^{0,1})} \\ &= -\sqrt{-1}\tilde{X}^{0,1} + \sqrt{-1}\,\overline{\tilde{X}^{0,1}} - \sqrt{-1}\nu(\tilde{X}^{0,1}) + \sqrt{-1}\,\overline{\nu(\tilde{X}^{0,1})} \\ &= J_0(\tilde{X}) - J_0(\nu(\tilde{X}^{0,1}) + \overline{\nu(\tilde{X}^{0,1})}) = J_0(\tilde{X}) - J_0\nu(\tilde{X}). \end{aligned}$$

It follows that

$$J = J_0(\mathrm{Id} - \mu)(\mathrm{Id} + \mu)^{-1} = (\mathrm{Id} + \mu)J_0(\mathrm{Id} + \mu)^{-1}. \tag{3.1.3}$$

From this, we can see that $J + J_0 = 2J_0(\mathrm{Id} + \mu)^{-1}$ is invertible and

$$\mu = (J + J_0)^{-1}(J_0 - J) = (J - J_0)(J + J_0)^{-1}, \tag{3.1.4}$$

where the second form is derived from the first by the use of the identities $J_0(J + J_0) = (J_0 + J)J$ and $J(J + J_0) = (J + J_0)J_0$. We conclude that the Cayley correspondence is given by (3.1.3) and (3.1.4).

Finally, let J now be any ACS such that $J + J_0$ is invertible and let μ be the endomorphism of V defined by (3.1.4). Then

$$\begin{gathered} J_0\mu = J_0(J + J_0)^{-1}(J_0 - J) = (J + J_0)^{-1}J(J_0 - J) = \\ (J + J_0)^{-1}(J - J_0)J_0 = -\mu J_0, \end{gathered}$$

and thus $\mu \in \mathrm{Ant}_{J_0}(V)$. Furthermore, $\mathrm{Id} + \mu = 2J(J + J_0)^{-1}$ and is therefore invertible. Thus, J is commensurable to J_0 exactly when $J + J_0$ is invertible.

3.1.2 An atlas

We now describe a certain atlas for $\mathcal{J}(V)$ which, for each $J \in \mathcal{J}(V)$, contains a chart around J and into $\mathrm{Ant}_J(V)$. Suppose $J_0 \in \mathcal{J}$ and let $\mathcal{A}_{J_0} = \{\mu \in \mathrm{Ant}_{J_0}(V) \ :$

$\mathrm{Id} + \mu$ is invertible}, which is open in $\mathrm{Ant}_{J_0}(V)$. Consider the smooth function

$$\mathcal{F}_{J_0} : \mathcal{A}_{J_0} \to \mathrm{End}(V),\ \mu \mapsto J, \tag{3.1.5}$$

where J is given by (3.1.3). Its image is exactly $\mathcal{B}_{J_0} = \{J \in \mathcal{J}(V) : J$ is commensurable to $J_0\}$. As we have seen, $\mathcal{B}_{J_0}$ can also be described as the set of ACS's J such that $J + J_0$ is invertible. Thus $\mathcal{B}_{J_0}$ is open in $\mathcal{J}(V)$. The function $\mathcal{F}_{J_0}$ is a homeomorphism onto $\mathcal{B}_{J_0}$ since it has the continuous inverse

$$\mathcal{G}_{J_0} : \mathcal{B}_{J_0} \to \mathrm{Ant}_{J_0}(V),\ J \mapsto \mu, \tag{3.1.6}$$

where μ is given by (3.1.4). Moreover, $\mathcal{G}_{J_1}\mathcal{F}_{J_0} : \mathcal{G}_{J_0}(\mathcal{B}_{J_0} \cap \mathcal{B}_{J_1}) \to \mathcal{G}_{J_1}(\mathcal{B}_{J_0} \cap \mathcal{B}_{J_1})$, where J_1 is another ACS, is clearly smooth. Thus $\{\mathcal{F}_J : J \in \mathcal{J}(V)\}$ is an atlas for $\mathcal{J}(V)$ as desired.

For $\mu \in \mathcal{A}_{J_0}$, we easily compute that $D\mathcal{F}_{J_0}|_\mu(\varphi) = [\varphi(\mathrm{Id} + \mu)^{-1}, J]$ where $J = \mathcal{F}_{J_0}(\mu)$. If φ is in $\ker D\mathcal{F}_{J_0}|_\mu$ then

$$\begin{aligned} 0 &= \varphi(\mathrm{Id} + \mu)^{-1} J - J\varphi(\mathrm{Id} + \mu)^{-1} \\ &= (\varphi J_0 - J\varphi)(\mathrm{Id} + \mu)^{-1} \\ &= -(J_0 + J)\varphi(\mathrm{Id} + \mu)^{-1}, \end{aligned}$$

showing that $\varphi = 0$. Thus $\mathcal{F}_{J_0}$ is an immersion into $\mathrm{End}(V)$.[1]

Remark 3.1.2. From the work we did in 3.1.1, we see that by sending $\mu \in \mathrm{Ant}_J(V)$ to $\mathrm{pr}^{1,0}\,\mu\,(\mathrm{pr}^{0,1}|_V)^{-1}$, we obtain a real isomorphism from $\mathrm{Ant}_J(V)$ to $\bigwedge^{0,1} V_J \otimes_{\mathbb{C}} V_J^{1,0}$. Thus $\mathcal{J}(V)$ admits around each point J a parametrization by $\bigwedge^{0,1} V_J \otimes_{\mathbb{C}} V_J^{1,0}$.

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[1] This incidentally shows that the smooth structure we have defined on $\mathcal{J}(V)$ is the same as the one it inherits as a linear subspace of $\mathrm{End}(V)$.

3.2 On a manifold

We now consider the space of ACS's on a compact manifold M, which we denote by $\mathcal{J}(M)$. This is unsurprisingly a more complicated object than the space of ACS's on a vector space. Still, it can be endowed with the structure of a real Fréchet manifold.[2] To do so, we proceed in an analogous manner to the case of a vector space. The arguments are identical, mutatis mutandis.

The space $\mathcal{J}(M)$ is of course a subset of the space of C^∞ sections of the vector bundle $\mathrm{End}(TM) \simeq T^*M \otimes TM$. The terminology introduced in 3.1 can naturally be adapted to this setting by recasting the definitions fiberwise. Thus, given $J \in \mathcal{J}(M)$, we can speak of an ACS J' commensurable to J and a J-antilinear section $\mu \in C^\infty(M, T^*M \otimes TM)$, the space of which is denoted by $\mathrm{Ant}_J(M)$. Let $J_0 \in \mathcal{J}(M)$. As in (3.1.2), there is a correspondence[3]

$$\left\{ \text{ACS's commensurable to } J_0. \right\} \longleftrightarrow \left\{ \begin{array}{l} \mu \in \mathrm{Ant}_{J_0}(M) \text{ and} \\ \mathrm{Id} + \mu \text{ is invertible fiberwise.} \end{array} \right\}. \quad (3.2.1)$$

It is given explicitly by the formulas (3.1.3) and (3.1.4) (with composition interpreted fiberwise). The ACS J is commensurable to J_0 if and only if $J + J_0$ is invertible fiberwise.

Equip $\mathcal{J}(M)$ with the subspace topology induced from the usual C^∞ topology on $C^\infty(M, T^*M \otimes TM)$. For $J_0 \in \mathcal{J}(M)$, the space $\mathrm{Ant}_{J_0}(M)$ is a Fréchet subspace of $C^\infty(M, T^*M \otimes TM)$ since it can be described as $C^\infty(M, E_{J_0})$, where E_{J_0} is the subbundle of $T^*M \otimes TM$ whose fiber at $p \in M$ is $\mathrm{Ant}_{J_0(p)}(T_pM)$. Let $\mathcal{A}_{J_0} \subseteq \mathrm{Ant}_{J_0}(M)$ be the open set of sections μ such that $\mathrm{Id} + \mu \in C^\infty(M, T^*M \otimes TM)$ is invertible fiberwise and let $\mathcal{B}_{J_0} \subseteq \mathcal{J}(M)$ be the open set of ACS's commensurable to

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[2]For a basic discussion on Fréchet manifolds, consult (Hamilton, 1982).

[3]The section Id is the section that sends $p \in M$ to the identity map on T_pM.

J_0. We define functions $\mathcal{F}_{J_0}$ and $\mathcal{G}_{J_0}$ in the same way as in (3.1.5) and (3.1.6). These functions are homeomorphisms between $\mathcal{A}_{J_0}$ and $\mathcal{B}_{J_0}$, and the transition functions of the family $\{\mathcal{F}_J : J \in \mathcal{J}(T^{1,0}M)\}$ are smooth in the Fréchet sense. Thus $\{\mathcal{F}_J : J \in \mathcal{J}(M)\}$ is an atlas for $\mathcal{J}(M)$, turning it into a Fréchet manifold. As a consequence of Remark 3.1.2, the bundle E_{J_0} is isomorphic to the bundle $\bigwedge^{0,1} T^*_{J_0}M \otimes_{\mathbf{C}} T^{1,0}_{J_0}M$ (seen as $\mathbf{R}$-vector bundles). Thus $\mathcal{J}(M)$ also admits around each point J a parametrization by $C^\infty(M, \bigwedge^{0,1} T^*_J M \otimes_{\mathbf{C}} T^{1,0}_J M)$.

Seeing $\mathcal{F}_J$ as a map into $C^\infty(M, T^*M \otimes TM)$, the differential $D\mathcal{F}_J|_\mu$ is injective for every $\mu \in \mathcal{A}_J$. So the inclusion $\mathcal{J}(M) \hookrightarrow C^\infty(M, T^*M \otimes TM)$ (which is smooth as a map between Fréchet manifolds) has an injective differential everywhere. The image of this differential at $J \in \mathcal{J}(M)$ coincides with the image of $D\mathcal{F}_J|_0$. One has $T_J C^\infty(M, T^*M \otimes TM) \simeq C^\infty(M, T^*M \otimes TM)$ canonically and $D\mathcal{F}_J|_0(\nu) = [\nu, J] = 2\nu J$. Thus $T_J\mathcal{J}M$ is canonically identified with $\mathrm{Ant}_J(M) \subseteq C^\infty(M, T^*M \otimes TM)$.

Remark 3.2.1. Suppose J_0 and J are two integrable ACS's and that J is commensurable to J_0. To distinguish among the various objects associated with J_0 and J we will make use of subscripts (e.g. $T^{1,0}_{J_0}$ is the holomorphic tangent bundle associated to J_0). Let λ be the element in $C^\infty(M, \bigwedge^{0,1} T^*_{J_0}M \otimes_{\mathbf{C}} T^{1,0}_{J_0}M)$ to which J corresponds. Let $\mathrm{pr}^{0,1}_{J_0}$ and $\mathrm{pr}^{0,1}_J$ be the projections of $\Omega^1_{\mathbf{C}}M$ onto $\Omega^{0,1}_{J_0}M$ and $\Omega^{0,1}_J M$ respectively. Then, on $C^\infty_{\mathbf{C}}(M)$, we have

$$\bar\partial_J = \bar\partial_{J_0} + \lambda,$$

where λ is considered as an operator which sends $f \in C^\infty_{\mathbf{C}}(M)$ to the contraction $\lambda \cdot f \in \Omega^{0,1}M$. Indeed, the above is equivalent to the statement that for all $f \in C^\infty_{\mathbf{C}}(M)$ and $Z \in C^\infty(M, T^{\mathbf{C}}M)$ we have

$$\bar\partial_J f(Z) = (\bar\partial_{J_0} + \lambda) f(Z).$$

Using the easy identities $\bar\partial_{J_0} f(Z) = \mathrm{pr}^{0,1}_{J_0}(Z) f$, $\bar\partial_J f(Z) = \mathrm{pr}^{0,1}_J(Z) f$, and $\lambda f(Z) =$

$\lambda \circ \mathrm{pr}_{J_0}^{0,1}(Z)f$, this amounts to saying that

$$\mathrm{pr}_J^{0,1}(Z) = (\mathrm{pr}_{J_0}^{0,1} + \lambda \circ \mathrm{pr}_{J_0}^{0,1})(Z),$$

which is readily deduced from (3.1.1).

CHAPTER IV

DEFORMATIONS

We will now develop a framework for the deformation theory of compact complex manifolds. The basic notion is that of infinitesimal deformation which is essentially a measure of instantaneous change when one "deforms" an integrable ACS.

4.1 The Maurer-Cartan condition

Let M be a compact C^∞ manifold of dimension $2n$ and J_0 an integrable complex structure on M, which will be the default ACS on M for the remainder of this chapter. Recall that the space of ACS's on M, $\mathcal{J}(M)$, is a Fréchet manifold parametrized around $J_0 \in \mathcal{J}(M)$ by an open subset (in the C^∞ topology) of $\mathcal{A}^1 T^{1,0}M = C^\infty(M, \bigwedge^{0,1} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M)$. Thus an ACS J sufficiently close to J_0 corresponds to a section $\lambda_J \in \mathcal{A}^1 T^{1,0}M$. We would like to derive a condition for the integrability of J which involves only λ_J. We define a bracket operator $[\cdot,\cdot]$ for this purpose.

Definition 4.1.1. Let $\varphi \otimes X$ and $\psi \otimes Y$ be local C^1 sections of $\bigwedge^{0,p} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M$ and $\bigwedge^{0,q} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M$ respectively, defined over the same domain. We define the bracket $[\varphi \otimes X, \psi \otimes Y]$ as the $T^{1,0}M$-valued $p+q$-form

$$[\varphi \otimes X, \psi \otimes Y] = (\varphi \wedge \psi) \otimes [X, Y],$$

where $[\cdot,\cdot]$ on the RHS is the standard Lie bracket. It can be verified that this bracket operator extends to a $\mathbf{C}$-bilinear operator between C^1 $T^{1,0}M$-valued differential forms.

Proposition 4.1.1. *Let* $\lambda \in \mathcal{A}^p T^{1,0}M$, $\xi \in \mathcal{A}^q T^{1,0}M$ *and* $\tau \in \mathcal{A}^r T^{1,0}M$, *with* $p, q, r \geq 0$. *Then,*

(i) $[\xi, \lambda] + (-1)^{pq}[\lambda, \xi] = 0$;

(ii) $\bar{\partial}[\lambda, \xi] = [\bar{\partial}\lambda, \xi] + (-1)^p[\lambda, \bar{\partial}\xi]$;

(iii) $(-1)^{pr}[[\lambda, \xi], \tau] + (-1)^{pq}[[\xi, \tau], \lambda] + (-1)^{qr}[[\tau, \lambda], \xi] = 0$.

■

We can now state an integrability condition of the form sought after.

Lemma 4.1.1. *Let* J *be an ACS sufficiently close to* J_0 *so that it is represented by a form* $\lambda_J \in \mathcal{A}^1 T^{1,0}M$. *Then* J *is integrable if and only if*

$$\bar{\partial}\lambda_J + \frac{1}{2}[\lambda_J, \lambda_J] = 0. \tag{4.1.1}$$

The above is called the Maurer-Cartan condition.

Proof. Recall that an ACS J is integrable if and only if it is formally integrable, i.e. $T_J^{0,1}M$ is closed under the Lie bracket. We will show that J satisfies the Maurer-Cartan condition if and only if it is formally integrable. To make the notation less cumbersome, we will suppress the subscript J in λ_J and simply write λ.

Since the claim to be proven is local, we can work in local holomorphic (with respect to J_0) coordinates $z^1, \ldots, z^n$. In these coordinates, we write

$$\lambda = \sum_{i,j=1}^{n} \lambda_i^j d\bar{z}^j \otimes \frac{\partial}{\partial z^i}.$$

Now, looking at (3.1.1), a local frame for $T_J^{0,1}M$ is

$$\frac{\partial}{\partial \bar{z}^i} + \lambda\left(\frac{\partial}{\partial \bar{z}^i}\right), \; i = 1, \ldots, n.$$

If J is formally integrable, then

$$\left[\frac{\partial}{\partial \overline{z}^i} + \lambda\left(\frac{\partial}{\partial \overline{z}^i}\right), \frac{\partial}{\partial \overline{z}^j} + \lambda\left(\frac{\partial}{\partial \overline{z}^j}\right)\right] \in T_J^{0,1}M$$

for $i, j = 1, \ldots, n$. Since $\left[\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right] = 0$, this means

$$\sum_{k=1}^{n} \frac{\partial \lambda_k^j}{\partial \overline{z}^i} \frac{\partial}{\partial z^k} - \sum_{k=1}^{n} \frac{\partial \lambda_k^i}{\partial \overline{z}^j} \frac{\partial}{\partial z^k} + \sum_{k,l=1}^{n} \left[\lambda_k^i \frac{\partial}{\partial z^k}, \lambda_l^j \frac{\partial}{\partial z^l}\right] = 0.$$

Now,

$$\sum_{k=1}^{n} \frac{\partial \lambda_k^j}{\partial \overline{z}^i} \frac{\partial}{\partial z^k} - \sum_{k=1}^{n} \frac{\partial \lambda_k^i}{\partial \overline{z}^j} \frac{\partial}{\partial z^k} = \sum_{k=1}^{n} \left(\frac{\partial \lambda_k^j}{\partial \overline{z}^i} - \frac{\partial \lambda_k^i}{\partial \overline{z}^j}\right) \frac{\partial}{\partial z^k} = 2\, \overline{\partial} \lambda \left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right)$$

and

$$\sum_{k,l=1}^{n} \left[\lambda_k^i \frac{\partial}{\partial z^k}, \lambda_l^j \frac{\partial}{\partial z^l}\right] = [\lambda, \lambda]\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right).$$

Thus

$$\left(2\, \overline{\partial} \lambda + [\lambda, \lambda]\right)\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right) \in T_J^{0,1}M.$$

But since J is close to J_0, $T_{J_0}^{1,0}M \cap T_J^{0,1}M = 0$ whereas $2\, \overline{\partial} \lambda + [\lambda, \lambda] \in \mathcal{A}^2 T_{J_0}^{1,0}M$. Thus

$$2\, \overline{\partial} \lambda + [\lambda, \lambda] = 0,$$

proving the claim. ∎

4.2 Infinitesimal deformations and Kodaira-Spencer classes

Let Diff(M) be the group of diffeomorphisms from M to itself. This is a Fréchet Lie group and its Lie algebra, i.e. the tangent space at Id $\in$ Diff(M), can be identified with $C^\infty(M, TM)$. Under this identification, the exponential map $\exp : C^\infty(M, TM) \to \mathrm{Diff}(M)$ has the property that, for $X \in C^\infty(M, TM)$, the map $M \times \mathbf{R} \ni (x, t) \mapsto \exp(tX)(x)$ is exactly the global flow on M generated by X.[1]

---

[1]See (Milnor, 1983) or (Banyaga, 1997) for the relevant definitions and results.

Consider the action of $\mathrm{Diff}(M)$ on $\mathcal{J}(M)$ given by $F \cdot J = DF \circ J \circ DF^{-1}$. By Proposition 1.2.1, two integrable ACS's J and J' define biholomorphic complex manifolds if and only if there is an $F \in \mathrm{Diff}(M)$ such that $J' = F \cdot J$. Let γ be a curve in $\mathcal{J}(M)$ starting at J_0, passing only through integrable ACS's. Such a curve is called a *one-parameter deformation of* J_0. Suppose we wish to find a measure of the "instantaneous rate of change" of the complex manifold structure on M defined by γ at $t = 0$. Since the complex manifold structure defined by an integrable ACS does not change under the action of $\mathrm{Diff}(M)$, this measure should be same for γ as for $\sigma \cdot \gamma$, where σ is a curve in $\mathrm{Diff}(M)$ with $\sigma(0) = \mathrm{Id}$. A natural choice then is $\gamma'(0) + V_{J_0}$, where V_{J_0} is the image of the infinitesimal action of $\mathrm{Diff}(M)$ on $T_{J_0}\mathcal{J}(M) = \mathrm{Ant}_{J_0}(M)$, i.e. the space of vectors tangent to curves of the form $\exp(tX) \cdot J_0$ where $X \in C^\infty(M, TM)$. We make the following definition:

Definition 4.2.1. An *infinitesimal deformation* is a class in $\mathrm{Ant}_{J_0}(M)/V_{J_0}$ of the form $\gamma'(0) + V_{J_0}$, where γ is a one-parameter deformation of J_0.

We now find an alternative and simpler description of V_{J_0}. Let $X \in C^\infty(M, TM)$ and put $F_t = \exp(tX)$. Recall that $\mathcal{J}(M)$ can be parametrized around J_0 by both $\mathrm{Ant}_{J_0}(M)$ and $\mathcal{A}^1 T^{1,0} M$. In the first parametrization, a coordinate map around J_0 is given by

$$\mathcal{G}_{J_0}(J) = (J + J_0)^{-1}(J_0 - J) = (J - J_0)(J + J_0)^{-1},$$

where the operations are performed fiberwise. Here $J \in \{J' \in \mathcal{J}(M) \; : \; J' + J_0 \text{ is invertible fiberwise}\}$. Thus, for t small,

$$\mathcal{G}_{J_0}(F_t \cdot J_0) = (F_t \cdot J_0 + J_0)^{-1}(J_0 - F_t \cdot J_0) = (F_t \cdot J_0 - J_0)(F_t \cdot J_0 + J_0)^{-1}$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{G}_{J_0}(F_t \cdot J_0) = -\frac{1}{2} J_0 \mathcal{L}_X J_0 = \frac{1}{2}(\mathcal{L}_X J_0) J_0,$$

where $\mathcal{L}$ denotes the Lie derivative. Let $\mathcal{F}_{J_0}$ be the inverse of $\mathcal{G}_{J_0}$. Considering $\mathcal{F}_{J_0}$ as a map into the Fréchet manifold $C^\infty(M, T^*M \otimes TM) \supseteq \mathcal{J}(M)$, its differential is given by

$$D\mathcal{F}_{J_0}|_{\mu}(\nu) = [\nu(\mathrm{Id} + \mu)^{-1}, J],$$

where $J = \mathcal{F}_{J_0}(\mu)$. Now $\mathcal{G}_{J_0}(J_0) = 0$ and

$$D\mathcal{F}_{J_0}|_0 \left(-\frac{1}{2} J_0 \mathcal{L}_X J_0\right) = -\mathcal{L}_X J_0.$$

In other words,

$$\left.\frac{d}{dt}\right|_{t=0} F_t \cdot J_0 = -\mathcal{L}_X J_0.$$

We have thus established:

Proposition 4.2.1. *The space V_{J_0} is exactly $\{\mathcal{L}_X J_0 \ : \ X \in C^\infty(M)\}$.* ■

Now we find to what V_{J_0} corresponds through the parametrization by $\mathcal{A}^1 T^{1,0} M$ of $\mathcal{J}(M)$ around J_0. Let $\mathrm{pr}^{1,0}$ and $\mathrm{pr}^{0,1}$ denote the projection maps of TM onto $T^{1,0}M$ and $T^{0,1}M$ respectively. Where it appears, the superscript $1,0$, respectively $0,1$, denotes the application of $\mathrm{pr}^{1,0}$, respectively $\mathrm{pr}^{0,1}$. Let $Y \in C^\infty(M, TM)$. We have $-\frac{1}{2} J_0 \mathcal{L}_X J_0(Y) = (\mathrm{pr}^{1,0}|_{TM})^{-1}[Y^{0,1}, X^{1,0}]^{1,0}$. Indeed,

$$\begin{aligned}
(\mathrm{pr}^{1,0}|_{TM})^{-1}[Y^{0,1}, X^{1,0}]^{1,0} &= 2\Re([Y^{0,1}, X^{1,0}]^{1,0}) \\
&= \frac{1}{2}\Re([Y, X] - \sqrt{-1}[Y, J_0(X)] \\
&\quad + \sqrt{-1}[J_0(Y), X] + [J_0(Y), J_0(X)])^{1,0} \\
&= \frac{1}{4}\{[Y, X] + [J_0(Y), J_0(X)] - \\
&\quad J_0([Y, J_0 X]) + J_0([J_0(Y), X])\} \\
&= \frac{1}{4}\Big\{[J_0(Y), J_0(X)] - J_0([J_0(Y), X]) - \\
&\quad J_0([Y, J_0(X)]) - [Y, X] + 2[Y, X] + \\
&\quad 2J_0([J_0(Y), X])\Big\} \\
&= -\frac{1}{2}\{[X, Y] - J_0([X, J_0(Y)])\} = -\frac{1}{2} J_0 \mathcal{L}_X J_0(Y),
\end{aligned}$$

where we have used the fact that the Nijenhuis tensor N^{J_0} vanishes and the identity $\mathcal{L}_X J_0(Y) = [X, J_0(Y)] - J_0([X,Y])$.[2] The passage from $\mathrm{Ant}_{J_0}(M)$ to $\mathcal{A}^1 T^{1,0}M$ is given by $\mu \mapsto \mathrm{pr}^{1,0}\,\mu\,(\mathrm{pr}^{0,1}|_{TM})^{-1}$. Thus $-\frac{1}{2}J_0\mathcal{L}_X J_0$ is sent to the element in $\mathcal{A}^1 T^{1,0}M$ which corresponds to the mapping $Z \mapsto [Z, X^{1,0}]^{1,0}$, $Z \in T^{0,1}M$. By writing this in coordinates, we see that this is exactly $\overline{\partial}\, X^{1,0}$. So we have established:

Proposition 4.2.2. *Through the parametrization by $\mathcal{A}^1 T^{1,0}M$ of $\mathcal{J}(M)$ around J_0, V_{J_0} corresponds to the space $\{\overline{\partial}\, Z \;:\; Z \in C^\infty(M, T^{1,0}M)\}$.* ■

Let γ be a one-parameter deformation of J_0. Let λ be the corresponding curve through the parametrization around J_0 by $\mathcal{A}^1 T^{1,0}M$. For $n \geq 1$, the n^{th} order Taylor expansion of $\lambda(t)$ has the form

$$\lambda(t) = \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + O(t^{n+1}). \tag{4.2.1}$$

Also, $\lambda(t)$ satisfies the Maurer-Cartan condition at every (sufficiently small) t since $\gamma(t)$ is integrable, i.e.

$$\overline{\partial}\,\lambda(t) + \frac{1}{2}[\lambda(t), \lambda(t)] = 0.$$

Substituting (4.2.1) into the above and collecting terms according to their orders, we are lead to the equations

$$\begin{gathered}\overline{\partial}\,\lambda_1 = 0\\ \overline{\partial}\,\lambda_2 + \frac{1}{2}[\lambda_1, \lambda_1] = 0\\ \vdots\\ \overline{\partial}\,\lambda_n + \frac{1}{2}\sum_{i=1}^{n-1}[\lambda_i, \lambda_{n-i}] = 0.\end{gathered}$$

Thus

$$\overline{\partial}\,\lambda_n + \frac{1}{2}\sum_{i=1}^{n-1}[\lambda_i, \lambda_{n-i}] = 0, \quad n \geq 1. \tag{4.2.2}$$

---

[2]See (Kobayashi et Nomizu, 1963, Proposition 3.5, ch. 1).

In particular $\overline{\partial}\lambda_1$ vanishes. Since $\lambda_1 = \lambda'(0)$, we conclude:

Proposition 4.2.3. *Through the parametrization by $\mathcal{A}^1 T^{1,0}M$ around J_0, an infinitesimal deformation corresponds to an element of the first cohomology group of the differential complex $\{(\mathcal{A}^p T^{1,0}M, \overline{\partial}_{0,p}) : -\infty < p < \infty\}$, i.e.* $\ker \overline{\partial}_{0,1} \,/\, \mathrm{im}\, \overline{\partial}_{0,0}$. ∎

By the Dolbeaut theorem (Theorem 1.3.2),

Proposition 4.2.4. *Let $\Theta T^{1,0}M$ be the sheaf of germs of holomorphic vector fields over M. Then,*

$$\frac{\ker \overline{\partial}_{0,p}}{\mathrm{im}\, \overline{\partial}_{0,p-1}} \simeq H^p(M, \Theta T^{1,0}M),$$

for $p \geq 1$, where $H^p(M, \Theta T^{1,0}M)$ is the p^{th} Čech cohomology group. ∎

Thus $\ker \overline{\partial}_{0,1} \,/\, \mathrm{im}\, \overline{\partial}_{0,1} \simeq H^1(M, \Theta T^{1,0}M)$. Classes in $H^1(M, \Theta T^{1,0}M)$ are called *Kodaira-Spencer classes*. They are in one-to-one correspondence with classes in $\mathrm{Ant}_{J_0}(M)/V_{J_0}$. A Kodaira-Spencer class is said to be *integrable* if its corresponding class in $\mathrm{Ant}_{J_0}(M)/V_{J_0}$ is an infinitesimal deformation. Thus a one-parameter deformation γ has both an associated infinitesimal deformation and an associated integrable Kodaira-Spencer class. Describing the set of all Kodaira-Spencer classes is the main objective of deformation theory.

CHAPTER V

THE THEOREM OF EXISTENCE

Let M be a compact C^∞ manifold of dimension $2n$ and J_0 a complex structure on M. The ACS J_0 will be the default complex structure on M throughout this chapter. We use the same notation as in the previous chapter.

Let θ be a Kodaira-Spencer class, i.e. an element in $H^1(M, \Theta T^{1,0}M)$. A basic question to ask is whether θ is integrable, that is if there is a one parameter deformation of J_0, γ, such that the infinitesimal deformation $\gamma'(0) + V_{J_0}$ corresponds to θ via the one-to-one correspondence described before. This question is in general not trivially answered.

Suppose γ is as desired and that λ is the corresponding curve in $\mathcal{A}^1 T^{1,0}M$. Then $[\lambda'(0)] = \theta$. Looking at (4.2.2), we know that $[\lambda'(0), \lambda'(0)]$ is element of $\operatorname{im} \overline{\partial}_{0,1}$. From property (ii) in Proposition 4.1.1, we see that the bracket operator descends into an operator

$$[\cdot, \cdot] : H^p(M, \Theta T^{1,0}M) \times H^q(M, \Theta T^{1,0}M) \to H^{p+q}(M, \Theta T^{1,0}M).$$

Clearly then $[[\lambda'(0)], [\lambda'(0)]] = 0$. Thus we have:

Proposition 5.0.5. *A necessary condition for $\theta \in H^1(M, \Theta T^{1,0}M)$ to be integrable is that $[\theta, \theta] = 0$.*

The above condition is called the *first obstruction*. From (4.2.2), we can derive an infinite number of obstructions and so we should expect that Kodaira-Spencer classes cannot always be integrated. It turns out however that if we ask that the second cohomology group $H^2(M, \Theta T^{1,0}M)$ vanishes so that the first obstruction is never an obstacle, every Kodaira-Spencer class can indeed be integrated. This is the content of the all-important theorem of existence.

Theorem 5.0.1 (Theorem of existence). *Suppose $H^2(M, \Theta T^{1,0}M) = 0$. Then every Kodaira-Spencer class $\theta \in H^1(M, \Theta T^{1,0}M)$ is integrable, i.e. there exists a one parameter deformation of J_0, γ, such that θ corresponds to $\gamma'(0) + V_{J_0}$.*

Following are examples of complex manifolds satisfying the condition of the theorem.

Example 5.0.1 (Riemann surfaces). Let M be a Riemann surface. Then as $\bigwedge^{0,2} T^{*\mathbf{C}}M$ is trivial, $H^2(M, \Theta T^{1,0}M) = 0$ by the Dolbeaut theorem (Theorem 1.3.2).

Example 5.0.2 (Blow-ups of $\mathbf{CP}^2$). Let $p_1, \ldots, p_m$ be distinct points of $\mathbf{CP}^2$. The manifold obtained by blowing up $\mathbf{CP}^2$ at p_1, then the resulting manifold at p_2, and so on, as well as $\mathbf{CP}^2$ itself satisfy the condition of theorem. Proving this would take us too far afield; the interested reader is referred to (Kodaira, 2005, p. 220).

In what follows we will prove the theorem of existence. The basic idea is to construct a formal power-series $\eta(t) = \sum_{i=1}^{\infty} \eta_i t^i$ that satisfies the equations (4.2.2), and hence the Maurer-Cartan condition, *formally* and such that $[\eta_1] = \theta$. One then shows that, for t small, $\eta(t)$ converges in some superspace of $\mathcal{A}^1 T^{1,0}M$. If $\tilde{\eta}$ denotes the function thus defined, a regularity argument shows that 1. $\tilde{\eta}(t)$ is smooth for t small and 2. $\tilde{\eta}$ is a smooth function of t. This entails that $\tilde{\eta}(t)$ is a smooth curve in $\mathcal{A}^1 T^{1,0}M$ satisfying the Maurer-Cartan condition with $\tilde{\eta}(0) = 0$ and $[\tilde{\eta}'(0)] = \theta$. The corresponding curve in $\mathcal{J}(M)$ is then a one parameter deformation of J_0 with the desired property.

We will also provide another, somewhat simpler, proof which does not use power series but the inverse function theorem in Banach spaces instead.

5.1 Proof with power series

5.1.1 Preliminaries

Recall that $\mathcal{A}^p T^{1,0}M = C^\infty(M, \bigwedge^{0,p} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M)$. Now let $\mathcal{A}^p_k T^{1,0}M = C^k(M, \bigwedge^{0,p} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M)$, with $0 \leq k \leq \infty$. In other words, $\mathcal{A}^p_k T^{1,0}M$ is the set of C^k $T^{1,0}M$-valued $(0, p)$-forms.

We introduce a Hermitian metric h on $T^{1,0}M$. We use h to naturally define a Hermitian metric on each bundle $\bigwedge^{q,p} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M$, which we also denote by h. We also define a Riemannian metric on M via the $\mathbf{R}$-vector bundle isomorphism between TM and $T^{1,0}M$ given by $X \mapsto X - \sqrt{-1}\, J_0|_x(X)$, for $x \in M$ and $X \in T_xM$. We then have access to the machinery of Hodge theory introduced in Chapter 2.

The following proposition and corollary will be essential. This is where the hypothesis that $H^2(M, \Theta T^{1,0}M) = 0$ becomes crucial.

Proposition 5.1.1. *The operator $G : \mathcal{A}^2 T^{1,0}M \to \mathcal{A}^2 T^{1,0}M$ is the inverse of the operator $\Delta^{\bar\partial} : \mathcal{A}^2 T^{1,0}M \to \mathcal{A}^2 T^{1,0}M$.*

Proof. Since $H^2(M, \Theta T^{1,0}M) = 0$, we have $\mathcal{H}^{0,2}_h(T^{1,0}M) = 0$ by the Hodge-Dolbeaut theorem (Theorem 2.3.2). Thus for $\varphi \in \mathcal{A}^2 T^{1,0}M$, Poisson's equation is just

$$\varphi = \Delta^{\bar\partial} G\varphi,$$

from which the statement follows. ∎

Corollary 5.1.1. *Suppose $\varphi \in \mathcal{A}^2 T^{1,0}M$ is such that $\bar\partial \varphi = 0$. Then*

$$\varphi = \bar\partial\, \bar\partial^* G\varphi.$$

Proof. We have

$$\varphi = \Delta^{\bar\partial} G\varphi = \bar\partial\, \bar\partial^* G\varphi + \bar\partial^* \bar\partial G\varphi = \bar\partial\, \bar\partial^* G\varphi + \bar\partial^* G\, \bar\partial\varphi = \bar\partial\, \bar\partial^* G\varphi.$$

■

We fix a finite atlas $\{(U_j, \zeta_j)\}$ of M and a subordinate smooth partition of unity $\{\rho_j\}$. This allows us to define various norms on the sections of the bundles $\bigwedge^{0,p} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M$ (see Section 2.1, Chapter 2). We will in particular be interested in Hölder norms. Let $k \geq 0$ and $0 < \alpha < 1$. We write $\mathcal{A}^p_{k,\alpha}T^{1,0}M$ instead of $C^{k,\alpha}(M, \bigwedge^{0,p} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M)$. The space $(\mathcal{A}^p_{k,\alpha}T^{1,0}M, ||\cdot||_{C^{k,\alpha}})$ is Banach.

The main motivation for working with Hölder norms is the Schauder estimate (Theorem 2.2.2 (a))

$$||\varphi||_{C^{k+2,\alpha}} \leq C||\Delta^{\bar\partial}\varphi||_{C^{k,\alpha}} \text{ if } \varphi \perp \mathcal{H}^{0,p}_h(T^{1,0}M), \tag{5.1.1}$$

where $\varphi \in \mathcal{A}^p T^{1,0}M$ and $C > 0$ is a constant independent of φ.

Lemma 5.1.1. *Fix $k \geq 0$ and $0 < \alpha < 1$. Then for $\varphi \in \mathcal{A}^p T^{1,0}M$*

$$||G\varphi||_{k+2,\alpha} \leq C_1\, ||\varphi||_{k,\alpha}\,,$$

where $C_1 > 0$ is a constant independent of φ.

Proof. The L^2 norm $||\cdot||_{L^2}$ is equivalent to the norm $||\cdot||_h$ induced by the inner product $\langle\cdot,\cdot\rangle_h$. Since $||H\varphi||_h \leq ||\varphi||_h$, it follows that there is a constant $K > 0$ such that

$$||H\varphi||_{L^2} \leq K\, ||\varphi||_{L^2}\,.$$

Since $\mathcal{H}^{0,p}_h(T^{1,0}M)$ is finite dimensional, the norms $||\cdot||_{L^2}$ and $||\cdot||_{C^{k,\alpha}}$ are equivalent on $\mathcal{H}^{0,p}_h(T^{1,0}M)$. Also, since M is compact, $||\varphi||_{L^2} \leq K'||\varphi||_{C^{k,\alpha}}$ for some constant $K' > 0$. Combing this information with the above inequality, we get

$$||H\varphi||_{C^{k,\alpha}} \leq K_1\, ||\varphi||_{C^{k,\alpha}}\,, \tag{5.1.2}$$

for some constant $K_1 > 0$. Now, from (2.3.2), $HG\varphi = 0$. Thus $G\varphi \perp \mathcal{H}_h^{0,p}(T^{1,0}M)$. From (5.1.1) and (5.1.2) it follows that

$$\begin{aligned} ||G\varphi||_{C^{k+2,\alpha}} &\leq ||\Delta^{\bar\partial} G\varphi||_{C^{k,\alpha}} \\ &= ||\varphi - H\varphi||_{C^{k,\alpha}} \\ &\leq ||\varphi||_{C^{k,\alpha}} + ||H\varphi||_{C^{k,\alpha}} \\ &\leq C_1||\varphi||_{C^{k,\alpha}}, \end{aligned}$$

for a constant $C_1 > 0$.

■

Finally, note that, looking at Definition 4.1.1, we have, for p and q fixed,

$$||[\varphi, \psi]||_{C^{k,\alpha}} \leq C_2||\varphi||_{C^{k+1,\alpha}}||\psi||_{C^{k+1,\alpha}}, \tag{5.1.3}$$

where $\varphi \in \mathcal{A}^p T^{1,0}M$ and $\psi \in \mathcal{A}^q T^{1,0}M$ and $C_2 > 0$ is a constant.

5.1.2 Construction of the formal power series

We fix $\theta \in H^1(M, \Theta T^{1,0}M)$ for the remainder of this chapter. We will construct a formal power series $\eta(t) = \sum_{i=0}^{\infty} \eta_i t^i$ with coefficients in $\mathcal{A}^1 T^{1,0}M$ satisfying the equations (4.2.2), and therefore the Maurer-Cartan condition, formally.[1] We put $\eta_0 = 0$ and let $\eta_1 \in \mathcal{A}^1 T^{1,0}M$ be the harmonic representative of θ; clearly $[\eta_1] = \theta$. Given a formal power series $\mu(t) = \sum_{i=0}^{\infty} \mu_i t^i$ let $\mu^i(t) = \sum_{j=1}^{i} \mu_j t^j$. If $\eta(t)$ is as desired, then for $i \geq 1$,

$$\bar\partial \eta^i(t) + \frac{1}{2}[\eta^{i-1}(t), \eta^{i-1}(t)] = \bar\partial \eta(t) + \frac{1}{2}[\eta(t), \eta(t)] \bmod i,$$

[1]The operators $\bar\partial$ and $[\cdot,\cdot]$ have obvious interpretations when applied to formal power series in $\mathcal{A}^1 T^{1,0}M[[t]]$.

where mod i indicates that we ignore all terms of order greater than i on both sides of the equality. Thus the problem at hand is equivalent to constructing $\eta(t)$ so that the equation

$$\bar{\partial}\,\eta^i(t) + [\eta^{i-1}(t), \eta^{i-1}(t)] = 0 \bmod i \tag{5.1.4}$$

is satisfied for all $i \geq 1$. We now do this. Having already chosen η_0 and η_1, we recursively define, for $i \geq 2$, $\eta_i = -\bar{\partial}^* G\varphi_i$, where $\varphi_i = \frac{1}{2}\sum_{0<k<i}[\eta_k, \eta_{i-k}]$ (i.e. φ_i is the coefficient of the term of order i in $\frac{1}{2}[\eta^{i-1}(t), \eta^{i-1}(t)]$). To show (5.1.4) holds for all $i \geq 1$, we proceed by induction. The base case $i = 1$ is trivial. Thus assume that (5.1.4) holds up to $i \geq 1$. We have

$$\varphi_{i+1}t^{i+1} = \bar{\partial}\,\eta^i(t) + \frac{1}{2}[\eta^i(t), \eta^i(t)] \bmod i+1.$$

Now

$$\bar{\partial}\,\varphi_{i+1}t^{i+1} = \frac{1}{2}\bar{\partial}[\eta^i(t), \eta^i(t)] = [\bar{\partial}\,\eta^i(t), \eta^i(t)] \bmod i+1,$$

where we have used properties (i) and (ii) in Proposition 4.1.1. We also have

$$\bar{\partial}\,\eta^i(t) + \frac{1}{2}[\eta^i(t), \eta^i(t)] = \bar{\partial}\,\eta^i(t) + \frac{1}{2}[\eta^{i-1}(t), \eta^{i-1}(t)] = 0 \bmod i.$$

Thus $\bar{\partial}\,\eta^i(t) = -\frac{1}{2}[\eta^i(t), \eta^i(t)] \bmod i$ and we find

$$\bar{\partial}\,\varphi_{i+1}t^{i+1} = -\frac{1}{2}[[\eta^i(t), \eta^i(t)], \eta^i(t)] \bmod i+1.$$

But by property (iii) of Proposition 4.1.1, $[[\eta^i(t), \eta^i(t)], \eta^i(t)] = 0$. Thus $\bar{\partial}\,\varphi_{i+1} = 0$. We set $\eta_{i+1} = -\bar{\partial}^* G\varphi_{i+1}$. By Corollary 5.1.1, $\bar{\partial}\,\eta_{i+1} = -\varphi_{i+1}$. We then have

$$\begin{aligned}\bar{\partial}\,\eta^{i+1}(t) + \frac{1}{2}[\eta^i(t), \eta^i(t)] &= \bar{\partial}\,\eta_{i+1}t^{i+1} + \bar{\partial}\,\eta^i(t) + \frac{1}{2}[\eta^i(t), \eta^i(t)] \\ &= -\varphi_{i+1}t^{i+1} + \varphi_{i+1}t^{i+1} = 0 \bmod i+1.\end{aligned}$$

This completes the induction. Therefore

$$\bar{\partial}\,\eta(t) + \frac{1}{2}[\eta(t), \eta(t)] = 0$$

formally.

5.1.3 Proof of convergence

We now show that, given $k \geq 2$ and $0 < \alpha < 1$, there is a $\varepsilon > 0$ such that $\eta(t)$ converges for $|t| < \varepsilon$ in the norm $||\cdot||_{C^{k,\alpha}}$.

For a power series $\varphi(t) = \sum_{i=0}^{\infty} \varphi_i t^i$ in $A^p T^{1,0} M[[t]]$, we define $||\varphi||_{C^{k,\alpha}}(t) \in \mathbf{R}[[t]]$ to be the series $\sum_{i=0}^{\infty} ||\varphi_i||_{C^{k,\alpha}} t^i$. Given two series $a(t) = \sum_{i=0}^{\infty} a_i t^i$ and $b(t) = \sum_{i=0}^{\infty} b_i t^i$ in $\mathbf{R}[[t]]$, the expression $a(t) \prec b(t)$ means $a_i \leq b_i$ for $0 \leq i < \infty$.

If $A(t)$ is a power series in $\mathbf{R}[[t]]$, convergent for $|t| < \varepsilon$, with

$$||\eta^i||_{C^{k,\alpha}}(t) \prec A(t) \tag{5.1.5}$$

for $0 \leq i < \infty$, then $\eta(t)$ converges for $|t| < \varepsilon$ in the norm $||\cdot||_{C^{k,\alpha}}$. Fix $k \geq 2$ and $0 < \alpha < 1$. We find $A(t)$ such that (5.1.5) holds. We put

$$A(t) = \frac{b}{16} \sum_{i=1}^{\infty} \frac{c^{i-1}}{i^2} t^i,$$

where $b, c > 0$ are to be determined. Since the radius of convergence of the series $\sum_{i=1}^{\infty} \frac{t^i}{i^2}$ is 1, the radius of convergence of $A(t)$ is $1/c$. Let

$$B(s) = \sum_{i=1}^{\infty} \frac{s^i}{i^2}.$$

We have

$$B(s)^2 = \sum_{i=2}^{\infty} \sum_{j+k=i} \frac{s^i}{j^2 k^2}.$$

Since

$$\sum_{j+k=i} \frac{1}{j^2 k^2} \leq 2 \sum_{\substack{j+k=i \\ j \leq k}} \frac{1}{j^2 k^2} \leq 2 \sum_{\substack{j+k=i \\ j \leq k}} \frac{4}{j^2 i^2} < \frac{8}{i^2} \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{4\pi^2}{3i^2} < \frac{16}{i^2},$$

then

$$B(s)^2 \prec 16 \sum_{i=1}^{\infty} \frac{s^i}{i^2} = 16 B(s).$$

We have $A(t) = \frac{b}{16c}B(ct)$, and so

$$A(t)^2 \prec \frac{b}{c}A(t). \tag{5.1.6}$$

We will show that by induction that (5.1.5) holds for appropriate choices of b and c. Since $\eta_0 = 0$, (5.1.5) clearly holds for $i = 0$. The first order term of $A(t)$ is $b/16$. We set b to a value sufficiently large so that $b/16 > ||\eta_1||_{C^{k+1}} \geq ||\eta_1||_{C^{k,\alpha}}$. Then (5.1.5) is satisfied for $i = 1$. Assume now that (5.1.5) holds for $1 \leq i \leq j$. For $l \geq 1$, let φ_l be the coefficient of the term of order l in $\frac{1}{2}[\eta^{l-1}(t), \eta^{l-1}(t)]$. Recall that, by construction,

$$\eta_l = -\overline{\partial}^* G\varphi_l.$$

Since $\overline{\partial}^*$ is the formal adjoint of the first-order differential operator $\overline{\partial}$,

$$||\overline{\partial}^* \psi||_{C^{k,\alpha}} \leq K_1||\psi||_{C^{k+1,\alpha}} \quad \psi \in \mathcal{A}^2T^{1,0}M$$

for a constant $K_1 > 0$. By (5.1.3),

$$||[\psi, \xi]||_{C^{k-1,\alpha}} \leq K_2||\psi||_{C^{k,\alpha}}||\xi||_{C^{k,\alpha}} \quad \psi, \xi \in \mathcal{A}^2T^{1,0}M$$

for a constant $K_2 > 0$. Using the above and Lemma 5.1.1 we find, for $\psi \in \mathcal{A}^2T^{1,0}M$,

$$||\overline{\partial}^* G\psi||_{C^{k,\alpha}} \leq K_1||G\psi||_{C^{k+1,\alpha}} \leq C_1K_1||\psi||_{C^{k-1,\alpha}}.$$

It follows that

$$||\eta_{j+1}||_{C^{k,\alpha}} \leq C_1K_1||\varphi_{j+1}||_{C^{k-1,\alpha}}. \tag{5.1.7}$$

Let $\xi^l(t) = [\eta^l(t), \eta^l(t)]$ for $1 \leq l < \infty$. Then

$$\begin{aligned} ||\varphi_{j+1}||_{C^{k-1,\alpha}}t^{j+1} &\prec \frac{1}{2}||\xi^l||_{C^{k-1,\alpha}}(t) \\ &\prec \frac{K_2}{2}||\eta^j||_{C^{k,\alpha}}(t)\cdot||\eta^j||_{C^{k,\alpha}}(t) \\ &\prec \frac{K_2}{2}A(t)^2 \prec \frac{K_2b}{2c}A(t), \end{aligned}$$

where we have used (5.1.6) and the inductive hypothesis. Combining what we have found, we get

$$||\eta_{j+1}||_{C^{k,\alpha}} t^{j+1} \prec \frac{C_1 K_1 K_2 b}{2c} A(t).$$

Setting $c = C_1 K_1 K_2/2$, we have $||\eta_{j+1}||_{C^{k,\alpha}} t^{j+1} \prec A(t)$. Now $||\eta^{j+1}||_{C^{k,\alpha}}(t) = ||\eta^j||_{C^{k,\alpha}}(t) + ||\eta_{j+1}||_{C^{k,\alpha}} t^{j+1}$. By the inductive hypothesis, $||\eta^j||_{C^{k,\alpha}}(t) \prec A(t)$ and thus

$$||\eta^{j+1}||_{C^{k,\alpha}}(t) \prec A(t),$$

which completes the induction. Thus we have shown that $\eta(t)$ converges in $(\mathcal{A}^1_{k,\alpha} T^{1,0}M, ||\cdot||_{C^{k,\alpha}})$ for $|t| < 1/c$. The function it defines in $|t| < 1/c$ is denoted by $\tilde{\eta}$.

Note that

$$\overline{\partial}\,\tilde{\eta}(t) + \frac{1}{2}[\tilde{\eta}(t), \tilde{\eta}(t)] = 0$$

for $|t| < 1/c$. We cannot immediately conclude that $\tilde{\eta}(t)$ is C^∞ for $|t| < 1/c$, i.e. $\tilde{\eta}(t) \in \mathcal{A}^1 T^{1,0}M$, or that $\tilde{\eta}$ depends smoothly on t therein since our choice of c depends on k and α. We need a more elaborate approach.

5.1.4 Regularity

Fix $k \geq 2$ and $0 < \alpha < 1$. We use the same notation as in the previous section. Recall that η_1 is the harmonic representative of $\theta \in H^1(M, \Theta T^{1,0}M)$. In particular, $\overline{\partial}^* \eta_1 = 0$. By construction, η_i for $i \geq 2$ is in the image of $\overline{\partial}^*$. Thus, for $t \in (-1/c, 1/c)$, $\overline{\partial}^* \tilde{\eta}(t) = 0$ and $\Delta^{\overline{\partial}} \tilde{\eta}(t) = \overline{\partial}^* \overline{\partial}\,\tilde{\eta}(t)$. Since $\overline{\partial}\,\tilde{\eta}(t) + \frac{1}{2}[\tilde{\eta}(t), \tilde{\eta}(t)] = 0$, we have

$$\Delta^{\overline{\partial}} \tilde{\eta}(t) + \frac{1}{2}\overline{\partial}^* [\tilde{\eta}(t), \tilde{\eta}(t)] = 0.$$

Now consider the complexification of the series $\eta(t)$ to a series in $\mathcal{A}^1 T^{1,0}M[[z]]$, which we denote by $\eta(z)$. The series $\eta(z)$ converges for $|z| < 1/c$ and moreover defines a complex analytic function valued in $\mathcal{A}^1_{k,\beta} T^{1,0}M$ with domain $|z| < 1/c$, which we denote again by $\tilde{\eta}$. For $|z| < 1/c$, one has $\partial\tilde{\eta}(z)/\partial\bar{z} = 0$ and from the above it should be clear that

$$\Delta^{\bar{\partial}}\tilde{\eta}(z) + \frac{1}{2}\bar{\partial}^*[\tilde{\eta}(z), \tilde{\eta}(z)] = 0.$$

Now let V_ε be the open disc of radius $0 < \varepsilon < 1/c$. Let J' be the standard ACS on $V_\varepsilon \subseteq \mathbf{C} \simeq \mathbf{R}^2$. We equip the C^∞ manifold $M \times V_\varepsilon$ with the ACS $J_0 \oplus J'$. Note that this ACS is integrable and thus $M \times V_\varepsilon$ becomes a complex manifold. Let $\mathrm{pr}_M : M \times V_\varepsilon \to M$ be the projection onto the first component. We define a section $\hat{\eta}$ of the pullback bundle $\mathrm{pr}_M^* \bigwedge^{0,1} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M$, by putting $\hat{\eta}(p,z) = \tilde{\eta}(z)(p)$ for $(p,z) \in M \times V_\varepsilon$.

Recall that we had selected a finite atlas of M, $\{(U_j, \zeta_j)\}$. In the sequel, it will be convenient to assume that U_j is compactly contained in the domain of the coordinate map ζ_j, which can certainly be arranged. In U_s, $\tilde{\eta}(z)$ has the local representation

$$\tilde{\eta}(z)(\zeta_s) = \sum_{i,j=1}^{n} \tilde{\eta}^j_{s,i}(\zeta_s, z)\, d\zeta_s^j \otimes \frac{\partial}{\partial \zeta_s^i}, \tag{5.1.8}$$

where the functions $\tilde{\eta}^j_{s,i}$ are holomorphic in z and C^k in ζ_s. We can take $\{\zeta_s^1, \ldots, \zeta_s^n, z\}$ as local holomorphic coordinates for $M \times V_\varepsilon$ in $U_s \times V_\varepsilon$ and the corresponding local representation of $\hat{\eta}$ is also given by the RHS above, provided we reinterpret the terms there suitably. We thus see that $\hat{\eta} \in C^k(M \times V_\varepsilon, \mathrm{pr}_M^* \bigwedge^{0,1} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} T^{1,0}M)$. Note that since the functions $\tilde{\eta}^j_{s,i}$ are holomorphic in z, $\mathcal{L}_{\partial/\partial\bar{z}}\hat{\eta} = 0$.

The operators $\Delta^{\bar{\partial}}$ and $\bar{\partial}[\cdot,\cdot]$ can naturally be extended to operators acting on C^∞, indeed C^2, sections of $\mathrm{pr}_M^* \mathcal{A}^1 T^{1,0}M$. Furthermore, retaining the same symbols to denote those extensions, one has

$$\Delta^{\bar{\partial}}\hat{\eta} + \frac{1}{2}\bar{\partial}^*[\hat{\eta}, \hat{\eta}] = 0,$$

and thus

$$\left(-\mathcal{L}_{\frac{\partial^2}{\partial z \partial \bar{z}}} + \Delta^{\bar{\partial}}\right)\hat{\eta} + \frac{1}{2}\bar{\partial}^*[\hat{\eta}, \hat{\eta}] = 0.$$

The operator $-\mathcal{L}_{\frac{\partial^2}{\partial z \partial \bar{z}}} + \Delta^{\bar{\partial}}$ is easily seen to be elliptic of order 2. In U_s, we may regard $\hat{\eta}$ as a function taking $\zeta_s(U_s) \times V_\varepsilon \subseteq \mathbf{C}^{n+1} \simeq \mathbf{R}^{2n+2}$ into $\mathbf{C}^{2n}$. Taking this point of view, $\frac{1}{2}\,\overline{\partial}^*[\hat{\eta}, \hat{\eta}]$ takes the form

$$\sum_{|\beta|=2} a_\beta(\zeta_s, \hat{\eta}) D^\beta \hat{\eta} + \sum_{|\beta|\le 1} a_\beta(\zeta_s, \hat{\eta}, D\hat{\eta}) D^\beta \hat{\eta}$$

in U_s, where each a_β is a $2n \times 2n$ matrix of polynomial functions. For any $\delta > 0$, we can shrink ε sufficiently so that $||\tilde{\eta}^j_{s,i}||_{C^1} < \delta$ for every i and j (this is because $\tilde{\eta}^j_{s,i}$ is identically 0 on $U_s \times \{0\}$ and $U_s \Subset \text{Dom } \zeta_s$). Hence the operator

$$\Lambda = -\mathcal{L}_{\frac{\partial^2}{\partial z \partial \overline{z}}} + \Delta^{\overline{\partial}} + \sum_{|\beta|=2} a_\beta(\zeta_s, \hat{\eta}) D^\beta$$

is elliptic in $U_s \times V_\varepsilon$ if ε is sufficiently small. Now since

$$\Lambda\hat{\eta} + \sum_{|\beta|\le 1} a_\beta(\zeta_s, \hat{\eta}, D\hat{\eta}) D^\beta \hat{\eta} = 0,$$

we may resort to a bootstrapping procedure: Since $\hat{\eta}$ is C^k, the functions $a_\beta(\zeta_s, \hat{\eta}, D\hat{\eta}) D^\beta \hat{\eta}$, $\beta \le 1$, as well as the coefficient functions of Λ are C^{k-1}. By Theorem 2.2.3, $\hat{\eta}$ is $C^{k-1+2} = C^{k+1}$ in $U_s \times V_\varepsilon$. Following the same argument, $\hat{\eta}$ is also C^{k+2}, ad infinitum. Therefore $\hat{\eta}$ is actually C^∞ in $U_s \times V_\varepsilon$. So there is a ε sufficiently small so that $\hat{\eta}$ is C^∞ in $M \times V_\varepsilon$.

The smoothness of $\hat{\eta}$ implies that $\tilde{\eta}^j_{s,i}$ is C^∞ in ζ_s. From that, we can in turn conclude that $\tilde{\eta}(z)$ is in $\mathcal{A}^1 T^{1,0} M$ for $|z| < \varepsilon$ and depends smoothly on z therein. Thus $\tilde{\eta}$ is a C^∞ map taking $(-\varepsilon, \varepsilon)$ into $\mathcal{A}^1 T^{1,0} M$. It is easy to see that the germ of $\tilde{\eta}$ at $t = 0$ does not depend on our choice of $k \ge 2$ or $0 < \alpha < 1$. Let $\kappa \in \mathcal{A}^1 T^{1,0} M$ be the harmonic representative of $\theta \in H^1(M, \Theta T^{1,0} M)$. For every $k \ge 2$, $\tilde{\eta}'(0) = \kappa$ in $(\mathcal{A}^k T^{1,0} M, ||\cdot||_{C^k})$. Thus $\tilde{\eta}'(0) = \kappa$ in $\mathcal{A}^1 T^{1,0} M$ (equipped with the C^∞ topology). So $\tilde{\eta}$ is a curve in $\mathcal{A}^1 T^{1,0} M$ with $\tilde{\eta}(0) = 0$ and $[\tilde{\eta}'(0)] = \theta$. If γ is the corresponding curve in $\mathcal{J}(M)$, then $\gamma(0) = J_0$ and $\gamma'(0) + V_{J_0}$ corresponds to θ. Thus the theorem of existence is proven.

5.2 Proof with the inverse function theorem

We present here yet another proof of the theorem of existence, due to Kuranishi (Kuranishi, 1965). This proof does away with power series and instead calls upon the inverse function theorem in Banach spaces. We will work with the objects introduced in 5.1.1, using the Sobolev norms $||\cdot||_{H^k}$ instead of Hölder norms as before. We need the L^2 estimates (Theorem 2.2.2 (b))

$$||\varphi||_{H^{k+2}} \leq C||\,\Delta^{\bar\partial}\varphi||_{H^k} \text{ if } \varphi \perp \mathcal{H}_h^{0,p}(T^{1,0}M), \tag{5.2.1}$$
$$||\,\Delta^{\bar\partial}\varphi||_{H^k} \leq C'||\varphi||_{H^{k+2}},$$

where $C, C' > 0$ are constants and $\varphi \in \mathcal{A}^p T^{1,0}M$. Following the proof of Lemma 5.1.1 nearly to the letter, we can prove the estimate

$$||G\varphi||_{H^{k+2}} \leq C_1||\varphi||_{H^k}, \tag{5.2.2}$$

for some constant $C_1 > 0$. Let $Q = G\,\bar\partial^* = \bar\partial^* G$. Since $\bar\partial^*$ is a first-order linear differential operator, one has $||\bar\partial^*\psi||_{H^k} \leq K||\psi||_{H^{k+1}}$ for a constant $K > 0$. In combination with the above we get

$$||Q\varphi||_{H^{k+1}} \leq C_2||\varphi||_{H^k}, \tag{5.2.3}$$

where $C_2 > 0$ is a constant and $\varphi \in \mathcal{A}^p T^{1,0}M$. Let $\mathcal{B}_k^p T^{1,0}M =$ $W^{k,2}(M, \bigwedge^{0,p} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M)$. Then $\mathcal{B}_k^p T^{1,0}M$ is the completion of $\mathcal{A}^p T^{1,0}M$ with respect to $||\cdot||_{H^k}$. By (5.2.1), $\Delta^{\bar\partial}$ can be extended to a bounded operator $\Delta^{\bar\partial}$: $\mathcal{B}_{k+2}^p T^{1,0}M \to \mathcal{B}_k^p T^{1,0}M$. Likewise, by (5.2.2), G may be extended to a bounded operator $G : \mathcal{B}_k^p T^{1,0}M \to \mathcal{B}_{k+2}^p T^{1,0}M$. Recall Proposition 5.1.1; by continuity, $G : \mathcal{B}_k^2 T^{1,0}M \to \mathcal{B}_{k+2}^2 T^{1,0}M$ is the inverse of $\Delta^{\bar\partial} : \mathcal{B}_{k+2}^2 T^{1,0}M \to \mathcal{B}_k^2 T^{1,0}M$. Clearly $\Delta^{\bar\partial}$ sends $\mathcal{A}_{k+2}^2 T^{1,0}M$ to $\mathcal{A}_k^2 T^{1,0}M$. Thus G sends $\mathcal{A}_k^2 T^{1,0}M$ to $\mathcal{A}_{k+2}^2 T^{1,0}M$.

Now we define

$$\Phi = \{\varphi \in \mathcal{A}^1 T^{1,0}M \;:\; \bar\partial\varphi + \frac{1}{2}[\varphi,\varphi] = 0 \text{ and } \bar\partial^*\varphi = 0\}.$$

Let $\varphi \in \Phi$. Since $\bar{\partial}^* \varphi = 0$, $\Delta^{\bar{\partial}} \varphi = \bar{\partial}^* \bar{\partial} \varphi$. Thus $\Delta^{\bar{\partial}} \varphi + \frac{1}{2} \bar{\partial}^* [\varphi, \varphi] = 0$. Applying Green's operator to this last equation, we get $G \Delta^{\bar{\partial}} \varphi + \frac{1}{2} Q[\varphi, \varphi] = 0$. Since $\varphi = H\varphi + \Delta^{\bar{\partial}} G\varphi$, we have $G \Delta^{\bar{\partial}} \varphi = \varphi - H\varphi$. Thus

$$\varphi + \frac{1}{2} Q[\varphi, \varphi] = H\varphi.$$

Define

$$\Psi = \{\varphi \in \mathcal{A}^1 T^{1,0} M \;:\; \varphi + \frac{1}{2} Q[\varphi, \varphi] \in \mathcal{H}_h^{0,1}(T^{1,0} M)\}.$$

The above then says $\Phi \subseteq \Psi$. Let $F : \mathcal{A}^1 T^{1,0} M \to \mathcal{A}^1 T^{1,0} M$ be the functional

$$F\varphi = \varphi + \frac{1}{2} Q[\varphi, \varphi].$$

From Definition 4.1.1, it is clear that

$$||[\varphi, \psi]||_{H^k} \leq ||\varphi||_{H^{k+1}} ||\psi||_{H^{k+1}} \tag{5.2.4}$$

for $\varphi, \psi \in \mathcal{A}^p T^{1,0} M$. This together with (5.2.3) gives us for $k \geq 2$

$$||F\varphi||_{H^k} \leq ||\varphi||_{H^k} + \frac{1}{2} ||Q[\varphi, \varphi]||_{H^k} \leq ||\varphi||_{H^k} + K' ||\varphi||_{H^k}^2$$

for a constant $K' > 0$. In particular, F extends continuously to a function $F : \mathcal{B}_k^1 T^{1,0} M \to \mathcal{B}_k^1 T^{1,0} M$. Since $[\cdot, \cdot]$ is bilinear and Q linear, we easily see that F is in fact complex analytic. Clearly $DF|_0 = \mathrm{Id}$ and thus F is invertible near 0. The local inverse, F^{-1}, is also complex analytic. Choose $\varepsilon > 0$ sufficiently small that $\{\varphi \in \mathcal{B}_k^1 T^{1,0} M \;:\; ||\varphi||_{H^k} < \varepsilon\}$ is in the domain of F^{-1}. Let $m < \infty$ be the dimension of $\mathcal{H}_h^{0,1}(T^{1,0} M)$. We may identify $(\mathcal{H}_h^{0,1}(T^{1,0} M), ||\cdot||_{H^k})$ with $\mathbf{C}^m$ equipped with the usual Euclidean norm. Set

$$W = \{z \in \mathbf{C}^m \;:\; ||z|| < \varepsilon\}.$$

The set W identifies with an open subset of $\mathcal{H}_h^{0,1}(T^{1,0} M)$ contained in Dom F^{-1}. For $z \in W$, let $\eta(z) = F^{-1}(z)$. Now fix $k > n + 2$. For $2 \leq k' < k - n$ then by the Sobolev embedding theorem $\eta(z) \in \mathcal{A}_{k'}^1 T^{1,0} M$ and we have

$$F\eta(z) = \eta(z) + \frac{1}{2} Q[\eta(z), \eta(z)] = z. \tag{5.2.5}$$

Now for $\varphi \in \mathcal{A}^2_1 T^{1,0}M$, $\Delta^{\bar\partial} Q\varphi = \Delta^{\bar\partial}\bar\partial^* G\varphi = \bar\partial^* \Delta^{\bar\partial} G\varphi = \bar\partial^* \varphi$. Furthermore, $z \in \mathcal{H}^{0,1}_h(T^{1,0}M)$ (with identification). Thus

$$\Delta^{\bar\partial}\eta(z) + \frac{1}{2}\bar\partial^*[\eta(z), \eta(z)] = 0.$$

Keeping in mind that $\eta : W \to \mathcal{B}^1_k T^{1,0}M$ is complex analytic, we can show that $\eta(z) \in \mathcal{A}^1 T^{1,0}M$ and η smoothly depends on z in a manner completely analogous to that in 5.1.4, replacing $M \times V_\varepsilon$ by $M \times W$, $\mathcal{L}_{\partial^2/\partial z\partial\bar z}$ by $\sum_{i=1}^m \mathcal{L}_{\partial^2/\partial z^i \partial\bar z^i}$ and $\tilde\eta$ by η. Note that the germ of η at $z = 0$ is independent of $k > n + 2$.

We would like to show that furthermore $\eta(z)$ satisfies the Maurer-Cartan condition for all $z \in W$. We will show that in fact $\eta(z) \in \Phi$. Note that since the image of $\eta(z)$ is $F^{-1}(W)$, it is an open neighborhood of Ψ in the $||\cdot||_{H^k}$ norm topology. Since it contains 0, it covers an open neighborhood of $\Phi \subseteq \Psi$. Since $\bar\partial^* Q = (\bar\partial^*)^2 G = 0$, (5.2.5) gives us $\bar\partial^* \eta(z) = 0$ (recall that $\bar\partial^* z = 0$ since z is harmonic, with identification). Since $\bar\partial z = 0$, we have $\bar\partial\eta(z) = -\frac{1}{2}\bar\partial Q[\eta(z), \eta(z)]$ by (5.2.5) again. Then

$$\begin{aligned} \bar\partial\eta(z) + \frac{1}{2}[\eta(z), \eta(z)] &= -\frac{1}{2}\bar\partial Q[\eta(z), \eta(z)] + \frac{1}{2}[\eta(z), \eta(z)] \\ &= \frac{1}{2}Q\,\bar\partial[\eta(z), \eta(z)] \end{aligned}$$

where we have used

$$(\mathrm{Id} - \bar\partial Q)\varphi = (\Delta^{\bar\partial} G - \bar\partial\bar\partial^* G)\varphi = (\bar\partial^*\bar\partial G)\varphi = Q\,\bar\partial\varphi,$$

for $\varphi \in \mathcal{A}^2 T^{1,0}M$. Thus the task reduces to showing that $Q\,\bar\partial[\eta(z), \eta(z)] = 0$. We have

$$\begin{aligned} Q\,\bar\partial[\eta(z), \eta(z)] &= 2Q[\bar\partial\eta(z), \eta(z)] \\ &= Q[-\bar\partial Q[\eta(z), \eta(z)], \eta(z)] \\ &= Q[Q\,\bar\partial[\eta(z), \eta(z)], \eta(z)] - \underbrace{Q[[\eta(z), \eta(z)], \eta(z)]}_{=0} \\ &= Q[Q\,\bar\partial[\eta(z), \eta(z)], \eta(z)] \end{aligned}$$

where we have used the fact that $[[\eta(z), \eta(z)], \eta(z)] = 0$, which follows from property (iii) of Proposition 4.1.1. By using (5.2.3) and (5.2.4), we then find

$$||Q\,\bar{\partial}[\eta(z), \eta(z)]||_{H^k} \leq K||Q\,\bar{\partial}[\eta(z), \eta(z)]||_{H^k}||\eta(z)||_{H^k}.$$

for some constant $K > 0$. If we then change ε so that $\varepsilon < 1/K$, then clearly the above leaves us no choice but to conclude that $Q\,\bar{\partial}[\eta(z), \eta(z)] = 0$. Thus $\eta(z) \in \Phi$ and in particular satisfies the Maurer-Cartan condition.

Fix $k > n + 2$ for now. We regard η as a smooth (in fact analytic) map $\eta : W \subseteq \mathcal{H}_h^{0,1}(T^{1,0}M) \to \mathcal{B}_k^1 T^{1,0}M$. The following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\ \iota\ } & \text{Dom } F^{-1} \\ & \searrow^{\eta} & \downarrow F^{-1} \\ & & \mathcal{B}_k^1 T^{1,0}M \end{array}$$

Since $DF^{-1}|_0 = \text{Id}$, we see that $D\eta|_0$ is, after canonical identifications, the inclusion $\mathcal{H}_h^{0,1}(T^{1,0}M) \hookrightarrow \mathcal{A}^1T^{1,0}M \subseteq \mathcal{B}_k^1T^{1,0}M$. Let κ be the harmonic representative of $\theta \in H^2(M, \Theta T^{1,0}M)$. Then there is a unique vector $\mathbf{c} = (c_1, \ldots, c_{2m}) \in \mathbf{R}^{2m} \simeq \mathbf{C}^m \simeq \mathcal{H}_h^{0,1}(T^{1,0}M)$ with

$$\sum_{i=1}^{2m} c_i \frac{\partial \eta}{\partial x^i}(0) = \kappa. \tag{5.2.6}$$

Since the inclusion of $\mathcal{B}_l^1T^{1,0}M \hookrightarrow \mathcal{B}_k^1T^{1,0}M$ is continuous for $l \geq k$ and the germ of η at 0 is independent of k, (5.2.6) is also independent of k. Given an integer $k' > n$ and k sufficiently large, the Sobolev embedding theorem tells us that $\mathcal{A}_{k'}^1T^{1,0}M \subseteq \mathcal{B}_k^1T^{1,0}M$ and that this inclusion is continuous. Thus (5.2.6) holds when seeing as η function valued in $\mathcal{A}_{k'}^1T^{1,0}M$ (equipped with the $C^{k'}$ topology), as well. We can then conclude that (5.2.6) holds when seeing η as a function valued in $\mathcal{A}^1T^{1,0}M$ (equipped with the C^∞ topology). For $t \in \mathbf{R}$ small, define $\eta(t) = \eta(t\mathbf{c})$. Clearly then $[\eta'(0)] = \theta$ and satisfies the Maurer-Cartan condition. The corresponding curve γ in $\mathcal{J}(M)$ is a one parameter deformation and $\gamma'(0) + V_{J_0}$ corresponds to θ, thus proving theorem of existence.

CHAPTER VI

THE TIAN-TODOROV THEOREM

The Tian-Todorov theorem states that for certain kinds of complex manifolds, namely those that can be turned into so-called Calabi-Yau manifolds, all Kodaira-Spencer classes are integrable.

Theorem 6.0.1 (Tian-Todorov). *Let (M, h) be a Calabi-Yau manifold. Then every Kodaira-Spencer class $\theta \in H^1(M, \Theta T^{1,0}M)$ is integrable.*

The relevant definitions will be given in the next section. Following this, we will prove the theorem in a way nearly identical to what was done in Chapter 5 to prove the theorem of existence by way of power series. The argument must be slightly modified however since $H^2(M, \Theta T^{1,0}M)$ does not necessarily vanish in this case.

6.1 Kähler geometry

We give here a very brief primer on Kähler geometry. We only need a few facts and definitions. Details and proofs can be found in (Griffiths et Harris, 1978), (Ballmann, 2006) or (Huybrechts, 2005). In the following, for Λ equal to ∂, $\overline{\partial}$, d or $\partial\,\overline{\partial}$, we will say that a differential form φ is Λ-closed if $\Lambda\varphi = 0$ and that it is Λ-exact if there exists a differential form ψ in the appropriate space such that $\Lambda\psi = \varphi$.

Suppose M is a complex manifold of complex dimension n and h a Hermitian metric on $T^{1,0}M$. By using the natural isomorphism between $T^{1,0}M$ and TM, we may think of h as Riemannian metric on M. If J is the almost complex structure corresponding to the complex manifold structure on M, the *fundamental 2-form* of h is the 2-form ω defined by

$$\omega(X,Y) = h(JX,Y),$$

where X and Y are real vector fields over M.

Definition 6.1.1. The Hermitian metric h is said to be *Kähler* if its fundamental 2-form ω is closed, i.e. $d\omega = 0$.

A manifold together with a Kähler metric is called a *Kähler manifold.* If moreover the manifold is compact and admits a non-vanishing holomorphic $(n,0)$-form, it is called a *Calabi-Yau manifold.* A fundamental property of manifolds which admit Kähler metrics is the following:

Lemma 6.1.1 ($\partial\bar{\partial}$-lemma, (Huybrechts, 2005, p. 128)). *Suppose that M is a compact complex manifold that admits a Kähler metric. Let $\varphi \in \Omega^{p,q}M$ be d-closed. The following are equivalent:*

- **(i)** *The form φ is d-exact.*
- **(ii)** *The form φ is ∂-exact.*
- **(iii)** *The form φ is $\bar{\partial}$-exact.*
- **(iv)** *The form φ is $\partial\bar{\partial}$-exact.*

Let (M,h) be a Kähler manifold. We use h to obtain, in a natural way, a Hermitian metric on the holomorphic vector bundle $\bigwedge^{p,0} T^{*\mathbb{C}}M$, which we denote by $\tilde{h}$. Next, we define a Hermitian metric on the bundle $\bigwedge^{0,q} T^{*\mathbb{C}}M$ in a natural way using h, which we also denote by h. We then equip the bundle $\bigwedge^{p,q} T^{*\mathbb{C}}M \simeq \bigwedge^{0,q} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} \bigwedge^{p,0} T^{*\mathbb{C}}M$

with the Hermitian metric $h \otimes \tilde{h}$. Finally, we equip M with its natural orientation and put on it the Riemannian metric g corresponding to h via the usual $\mathbf{R}$-vector bundle isomorphism between TM and $T^{1,0}M$. This done, we may use the tools of Hodge theory.[1] The fact that h is Kähler has the benefit of ensuring the properties that are listed in the following proposition.

Proposition 6.1.1. *We have*

(i) *The operator* $\Delta^{\bar\partial}$ *commutes with* ∂.

(ii) *If* $\varphi \in \Omega^{p,q}M$ *is harmonic, then* $\partial\varphi = 0$.

(iii) *We have* $\partial\,\overline{\partial}^* + \overline{\partial}^*\,\partial = 0$.

■

6.2 Preliminaries

We use here the same notation as in Chapter 4. Suppose M is a compact complex manifold of dimension n and h a Kähler metric on M such that (M,h) is a Calabi-Yau manifold. Then M admits a non-vanishing holomorphic $(n,0)$-form, say ξ. For $0 \le p \le n$, we define a $\mathbf{C}$-vector bundle isomorphism

$$\Phi : \bigwedge^{p} T^{1,0}M \to \bigwedge^{n-p,0} T^{*\mathbf{C}}M,$$

as follows: if $V_1 \wedge \cdots \wedge V_p \in \bigwedge^p T^{1,0}_x M$, then $\Phi_x(V_1 \wedge \cdots \wedge V_p) \in \bigwedge^{n-p,0} T^{*\mathbf{C}}_x M$ is the $(n-p,0)$-form at x defined by $\Phi_x(V_1 \wedge \ldots \wedge V_p)(W_1,\ldots,W_{n-p}) = \xi_x(V_1,\ldots,V_p,W_1,\ldots,W_{n-p})$ for $W_1,\ldots,W_{n-p} \in T^{\mathbf{C}}_x M$. We then extend Φ by linearity; it is a simple matter to show that Φ is indeed a $\mathbf{C}$-vector bundle isomorphism. Passing

---

[1] Note that we regard $\bigwedge^{p,q} T^{*\mathbf{C}}M$ as the vector bundle of $(0,q)$-forms with values in the holomorphic vector bundle $\bigwedge^{p,0} T^{*\mathbf{C}}M$.

through the isomorphism (1.1.4), we obtain another **C**-vector bundle isomorphism

$$\mathrm{Id}\otimes\Phi : \bigwedge^{0,q} T^{*\mathbf{C}}M \otimes_{\mathbf{C}} \bigwedge^{p} T^{1,0}M \to \bigwedge^{n-p,q} T^{*\mathbf{C}}M.$$

The isomorphism $\mathrm{Id}\otimes\Phi$ induces a $C^\infty_{\mathbf{C}}$-module isomorphism

$$P : \mathcal{A}^q \bigwedge^{p} T^{1,0}M \to \Omega^{n-p,q}M.$$

We define an operator

$$L : \mathcal{A}^q \bigwedge^{p} T^{1,0}M \to \mathcal{A}^q \bigwedge^{p-1} T^{1,0}M$$

by putting $L = P^{-1}\partial P$.

Proposition 6.2.1. *We have* $\overline{\partial}\, P = P\,\overline{\partial}$ *and* $\overline{\partial}\, L = -L\,\overline{\partial}$.

Proof. Let $z^1, \dots, z^n$ be local holomorphic coordinates on M. Since ξ is holomorphic, we have $\xi = f dz^1 \wedge \cdots \wedge dz^n$ in U, where f is a holomorphic function. Let $A = \{a_1, \dots, a_k\} \subseteq \{1, \dots, n\}$. For short hand, dz^A, respectively $d\overline{z}^A$, stands for $dz^{a_1} \wedge \cdots \wedge dz^{a_k}$, respectively $d\overline{z}^{a_1} \wedge \cdots \wedge d\overline{z}^{a_k}$. Likewise, $\frac{\partial}{\partial z^A}$ stands for $\frac{\partial}{\partial z^{a_1}} \wedge \cdots \wedge \frac{\partial}{\partial z^{a_k}}$.

Now for $|I| = q$ and $|J| = q$,

$$P\left(g d\overline{z}^I \otimes \frac{\partial}{\partial z^J}\right) = (-1)^{(\sum_{k\in J} k)+p} f g d\overline{z}^I \otimes dz^{\{1,\dots,n\}\setminus J}$$

and therefore

$$\begin{aligned}\overline{\partial}\, P\left(g d\overline{z}^I \otimes \frac{\partial}{\partial z^J}\right) &= (-1)^{(\sum_{k\in J} k)+p} f(\overline{\partial}\, g) d\overline{z}^I \otimes dz^{\{1,\dots,n\}\setminus J} \\ &= P\left(\overline{\partial}\, g d\overline{z}^I \otimes \frac{\partial}{\partial z^J}\right),\end{aligned}$$

where we have used (1.2.1) and Proposition 1.2.4. Therefore $\overline{\partial}\, P = P\,\overline{\partial}$. So we find

$$\overline{\partial}\, L = \overline{\partial}\, P^{-1}\partial P = P^{-1}\,\overline{\partial}\,\partial P = -P^{-1}\partial\,\overline{\partial}\, P = -P^{-1}\partial P\,\overline{\partial} = -L\,\overline{\partial}.$$

∎

Given $\varphi \otimes X \in \mathcal{A}^{q_1} \bigwedge^{p_1} T^{1,0}M$ and $\psi \otimes Y \in \mathcal{A}^{q_2} \bigwedge^{p_2} T^{1,0}M$, we define

$$(\varphi \otimes X) \wedge (\psi \otimes Y) = (-1)^{p_1 \cdot q_2}(\varphi \wedge \psi) \otimes (X \wedge Y).$$

We can then extend $\wedge$ to an operator

$$\wedge : \mathcal{A}^{q_1} \bigwedge^{p_1} T^{1,0}M \times \mathcal{A}^{q_2} \bigwedge^{p_2} T^{1,0}M \to \mathcal{A}^{q_1+q_2} \bigwedge^{p_1+p_2} T^{1,0}M.$$

The following lemma is the real workhorse of our proof of the Tian-Todorov theorem.

Lemma 6.2.1 (Tian-Todorov lemma). *Let* $\alpha \in \mathcal{A}^p T^{1,0}M$ *and* $\beta \in \mathcal{A}^q T^{1,0}M$. *We have*

$$(-1)^p[\alpha, \beta] = L(\alpha \wedge \beta) - (L\alpha) \wedge \beta + (-1)^p \alpha \wedge L\beta.$$

Proof. Since both $[\cdot, \cdot]$ and $\wedge$ are bilinear operations, and that the above statement is really a local one, it suffices to prove it for $\alpha = a d\bar{z}^I \otimes \frac{\partial}{\partial z^i}$ and $\beta = b d\bar{z}^J \otimes \frac{\partial}{\partial z^j}$ where $z^1, \ldots, z^n$ are local holomorphic coordinates on M.

Let $K(\alpha, \beta) = L(\alpha \wedge \beta) - L(\alpha) \wedge \beta + (-1)^p \alpha \wedge L(\beta)$. We have

$$\begin{aligned}
L(\alpha \wedge \beta) &= P^{-1}\partial P(\alpha \wedge \beta) \\
&= (-1)^q P^{-1}\partial P\left((d\bar{z}^I \wedge d\bar{z}^J) \otimes \left(a\frac{\partial}{\partial z^i} \wedge b\frac{\partial}{\partial z^j}\right)\right) \\
&= (-1)^q P^{-1}\partial\left((d\bar{z}^I \wedge d\bar{z}^J) \wedge P\left(a\frac{\partial}{\partial z^i} \wedge b\frac{\partial}{\partial z^j}\right)\right) \\
&= (-1)^p P^{-1}\left((d\bar{z}^I \wedge d\bar{z}^J) \wedge \partial P\left(a\frac{\partial}{\partial z^i} \wedge b\frac{\partial}{\partial z^j}\right)\right) \\
&= (-1)^p (d\bar{z}^I \wedge d\bar{z}^J) \otimes L\left(a\frac{\partial}{\partial z^i} \wedge b\frac{\partial}{\partial z^j}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
L(\alpha) \wedge \beta &= P^{-1}\partial P\left(d\bar{z}^I \otimes a\frac{\partial}{\partial z^i}\right) \wedge \beta \\
&= (-1)^p \left(d\bar{z}^I \otimes L\left(a\frac{\partial}{\partial z^i}\right)\right) \wedge \beta \\
&= (-1)^p (d\bar{z}^I \wedge d\bar{z}^J) \otimes \left(L\left(a\frac{\partial}{\partial z^i}\right) \wedge b\frac{\partial}{\partial z^j}\right).
\end{aligned}$$

By the same token,

$$(-1)^{p+1}\alpha \wedge L(\beta) = (d\bar{z}^I \wedge d\bar{z}^J) \otimes \left(a\frac{\partial}{\partial z^i} \wedge L\left(b\frac{\partial}{\partial z^j}\right)\right).$$

For short hand, given $\varphi \in \mathcal{A}^{p_1}T^{1,0}M$ and $\psi \in \mathcal{A}^{p_2}T^{1,0}M$, we put $K(\varphi, \psi) = L(\varphi \wedge \psi) - L(\varphi) \wedge \psi + (-1)^p \varphi \wedge L(\psi)$. Combining all of the preceding, we get

$$K(\alpha, \beta) = (-1)^p (d\bar{z}^I \wedge d\bar{z}^J) \otimes K\left(a\frac{\partial}{\partial z^i}, b\frac{\partial}{\partial z^j}\right).$$

Since $[\alpha, \beta] = (d\bar{z}^I \wedge d\bar{z}^J) \otimes \left[a\frac{\partial}{\partial \bar{z}^i}, b\frac{\partial}{\partial \bar{z}^j}\right]$ it suffices to show that

$$K\left(a\frac{\partial}{\partial z^i}, b\frac{\partial}{\partial z^j}\right) = \left[a\frac{\partial}{\partial \bar{z}^i}, b\frac{\partial}{\partial \bar{z}^j}\right]. \tag{6.2.1}$$

Notice that, for $A \subseteq \{1, \dots, n\}$, we have

$$P\left(\frac{\partial}{\partial z^A}\right) = (-1)^{(\sum_{i\in A} i)+|A|} f d\bar{z}^{\{1,\dots,n\}\setminus A}.$$

Thus

$$\begin{aligned} L\left(a\frac{\partial}{\partial z^i}\right) &= (-1)^{i+1} P^{-1} \partial P\left(af dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge dz^n\right) \\ &= P^{-1}\left(\frac{\partial(af)}{\partial z^i} dz^1 \wedge \cdots \wedge dz^n\right) = f^{-1}\frac{\partial(af)}{\partial z^i}, \end{aligned}$$

where $\widehat{dz^i}$ indicates that dz^i is omitted. Likewise,

$$L\left(b\frac{\partial}{\partial z^j}\right) = f^{-1}\frac{\partial(af)}{\partial z^j}.$$

Finally,

$$\begin{aligned} L\left(a\frac{\partial}{\partial z^i} \wedge b\frac{\partial}{\partial z^j}\right) &= (-1)^{i+j} P^{-1} \partial\left(abf dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge \cdots \wedge \widehat{dz^j} \wedge dz^n\right) \\ &= (-1)^{j+1} P^{-1}\Bigg(\frac{\partial(abf)}{\partial z^i} dz^1 \wedge \cdots \wedge \widehat{dz^j} \wedge dz^n + \\ &\qquad (-1)^i \frac{\partial(abf)}{\partial z^j} dz^1 \wedge \cdots \wedge \widehat{dz^i} \wedge dz^n\Bigg) \\ &= \frac{\partial(abf)}{\partial z^i} f^{-1} \frac{\partial}{\partial z^j} - \frac{\partial(abf)}{\partial z^j} f^{-1} \frac{\partial}{\partial z^i} \\ &= \left[a\frac{\partial}{\partial z^i}, b\frac{\partial}{\partial z^j}\right] + L\left(a\frac{\partial}{\partial z^i}\right) b\frac{\partial}{\partial z^j} - a\frac{\partial}{\partial z^i} L\left(b\frac{\partial}{\partial z^j}\right), \end{aligned}$$

and thus (6.2.1) holds. This completes the proof. ∎

Corollary 6.2.1. *Let $\alpha \in \mathcal{A}^{q_1}T^{1,0}M$ and $\beta \in \mathcal{A}^{q_2}T^{1,0}M$. If α and β are both ∂-closed then $P[\alpha,\beta]$ is ∂-exact.*

Proof. Set $\gamma = (-1)^{q_1}P(\alpha \wedge \beta)$. Then a simple computation invoking the previous lemma shows that $\partial\gamma = P[\alpha,\beta]$. ■

6.3 Proof of the Tian-Todorov theorem

We can now prove the Tian-Todorov theorem. As was already said, we very closely follow the proof of the theorem of existence with power series presented in Chapter 5. We retain the notation as well as the objects from the previous section, so that (M,h) is a Calabi-Yau manifold. Our starting point is the following: we have a Kodaira-Spencer class $\theta \in H^1(M,\Theta T^{1,0}M)$ and we must show that it is integrable.

As was done in the paragraph preceding Proposition 6.1.1, we equip M with its natural orientation and use h to obtain a Hermitian metric h on $\bigwedge^{0,q}T^{*\mathbf{C}}$, a Hermitian metric $\tilde{h}$ on $\bigwedge^{p,0}T^{*\mathbf{C}}M$ and a Riemannian metric g on M. We then equip the holomorphic vector bundle $T^{1,0}M$ with a metric h' defined as the pullback of $\tilde{h}$ through the $\mathbf{C}$-vector bundle isomorphism

$$\Phi : T^{1,0}M \to \bigwedge^{n-1,0}T^{*\mathbf{C}}M.$$

We emphasize that h' is *not* the Hermitian metric naturally induced on $T^{1,0}M$ by h, but rather it is a Hermitian metric whose definition depends on the isomorphism Φ and hence on the holomorphic $(n,0)$-form ξ chosen earlier.[2] We then equip the bundle

[2]Let h'' be the Hermitian metric on $T^{1,0}M$ naturally induced by h. In the case h is a *Kähler-Einstein* metric, then actually $h' = ch''$ for a constant $c \in \mathbf{C}$. This is a consequence of the fact that a holomorphic $(n,0)$-form is parallel with respect to the Levi-Cevita connection of a Kähler-Einstein metric. For this reason, when proving the Tian-Todorov theorem, most authors take h to be Kähler-Einstein and define $h' = h''$. However, as our approach shows, this is not necessary. See e.g. (Huybrechts, 2005) for the relevant definitions and results.

$\bigwedge^{0,q} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M$ with the Hermitian metric $h \otimes h'$. Thus we can employ Hodge theory for both the bundles $\bigwedge^{p,q} T^{*\mathbb{C}}M$ (equipped with the Hermitian metric $h \otimes \tilde{h}$) and the bundles $\bigwedge^{0,q} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M$ (equipped with the Hermitian metric $h \otimes h'$).

Note that $h \otimes h'$ is nothing more than the pullback of $h \otimes \tilde{h}$ via the vector bundle isomorphism

$$\mathrm{Id} \otimes \Phi : \bigwedge^{0,q} T^{*\mathbb{C}}M \otimes_{\mathbb{C}} T^{1,0}M \to \bigwedge^{n-1,q} T^{*\mathbb{C}}M.$$

Thus, because of the way P was defined, we have $\langle \alpha, \beta \rangle_{h\otimes h'} = \langle P\alpha, P\beta \rangle_{h\otimes\tilde{h}}$ for $\alpha, \beta \in \mathcal{A}^q T^{1,0}M$. In following we use subscripts to emphasize which metric is being used. For $\kappa \in \mathcal{A}^{q+1}$ one has

$$\begin{aligned}
\left\langle \alpha, \bar{\partial}^*_{h\otimes h'} \kappa \right\rangle_{h\otimes h'} &= \left\langle \bar{\partial}\,\alpha, \kappa \right\rangle_{h\otimes h'} \\
&= \left\langle P\,\bar{\partial}\,\alpha, P\kappa \right\rangle_{h\otimes\tilde{h}} \\
&= \left\langle \bar{\partial}\,P\alpha, P\kappa \right\rangle_{h\otimes\tilde{h}} \\
&= \left\langle P\alpha, \bar{\partial}^*_{h\otimes\tilde{h}}\,P\kappa \right\rangle_{h\otimes\tilde{h}} \\
&= \left\langle \alpha, P^{-1}\,\bar{\partial}^*_{h\otimes\tilde{h}}\,P\kappa \right\rangle_{h\otimes h'}
\end{aligned}$$

and so $\bar{\partial}^*_{h\otimes\tilde{h}}\,P = P\,\bar{\partial}^*_{h\otimes h'}$. This then implies that $\Delta^{\bar{\partial}}{}_{h\otimes\tilde{h}}\,P = P\,\Delta^{\bar{\partial}}{}_{h\otimes h'}$. In particular P sends harmonic forms to harmonic forms.

In 5.1.2, we constructed a formal power series $\eta(t) \in \mathcal{A}^1 T^{1,0}M[[t]]$ that satisfies the Maurer-Cartan condition formally. We do this again, slightly modifying the construction. As before we set $\eta_0 = 0$ and choose η_1 to be the harmonic representative of θ. For $i \geq 2$, we recursively define $\eta_i = -P^{-1}\,\bar{\partial}^*\,GP\varphi_i$, where $\varphi_i = \frac{1}{2}\sum_{0<k<i}[\eta_k, \eta_{i-k}]$. To show that $\eta(t)$ formally satisfies the Maurer-Cartan condition, it suffices to show that $\bar{\partial}\,\eta_i = -\varphi_i$: the proof of that fact given in 5.1.2 will then translate to the present setting. We show this below. Set $\varphi_1 = 0$.

Proposition 6.3.1. *For $i \geq 1$, $\partial \eta_i = -\varphi_i$, φ_i is $\overline{\partial}$-closed and $P\eta_i$ is ∂-closed.*

Proof. We proceed by induction. First since η_1 is harmonic, $\overline{\partial}\,\eta_1 = 0$ and so $\overline{\partial}\,P\eta_1 = P\,\overline{\partial}\,\eta_1 = 0$. Moroever, $P\eta_1$ is harmonic which then implies that it is ∂-closed by item (ii) of Proposition 6.1.1. Obviously $\overline{\partial}\,\varphi_1 = 0$. This takes care of the base case.

Assume now the statement of the proposition holds for $\eta_1, \ldots, \eta_{i-1}$ where $i \geq 2$. By Corollary 6.2.1 then, $P\varphi_i = \frac{1}{2}P\sum_{0<k<i}[\eta_k, \eta_{i-k}]$ is ∂-exact. Let γ_i be a differential form in the appropriate space such that $\partial\gamma_i = P\varphi_i$. Then

$$P\eta_i = -\overline{\partial}^* G\partial\gamma_i = \partial\,\overline{\partial}^* G\gamma_i,$$

where we have used Proposition 2.3.1 in combination with (iii) of Proposition 6.1.1. Thus $P\eta_i$ is ∂-exact and in particular ∂-closed. Next we show that $\overline{\partial}\,\varphi_i = 0$. In the case of $i = 2$, this is a consequence of the fact that $\overline{\partial}\,\eta_1 = 0$ and item (ii) of Lemma 4.1.1. For $i \geq 3$, we have

$$\begin{aligned}
\overline{\partial}\,\varphi_i &= \frac{1}{2}\sum_{0<j<i}[\overline{\partial}\,\eta_j, \eta_{i-j}] + [\eta_j, \overline{\partial}\,\eta_{i-j}] \\
&= -\frac{1}{2}\sum_{0<j<i}\left(\sum_{0<k<j}[[\eta_k, \eta_{j-k}], \eta_{i-j}] + \sum_{0<l<i-j}[\eta_j, [\eta_l, \eta_{i-j-l}]]\right) \\
&= -\frac{1}{2}\sum_{0<j<i}\sum_{0<k<j}[[\eta_k, \eta_{j-k}], \eta_{i-j}] + \frac{1}{2}\sum_{0<j<i}\sum_{0<l<j}[\eta_{i-j}, [\eta_l, \eta_{j-l}]] = 0
\end{aligned}$$

by items (i) and (ii) of 4.1.1.

Finally we show $\partial\eta_i = -\varphi_i$. As already shown, $P\varphi_i$ is ∂-exact. It is also $\overline{\partial}$-closed since φ_i is. By the $\partial\,\overline{\partial}$-lemma then, $P\varphi_i$ is $\overline{\partial}$-exact as well. By Corollary 2.3.1 then, $HP\varphi_i = 0$ and so $P\varphi_i = \Delta^{\overline{\partial}} GP\varphi_i$ by Poisson's equation. Thus

$$P\,\overline{\partial}\,\eta_i = -\overline{\partial}\,\overline{\partial}^* GP\varphi_i = -(\Delta^{\overline{\partial}} - \overline{\partial}^*\,\overline{\partial})GP\varphi_i = -P\varphi_i + \overline{\partial}^* G\,\overline{\partial}\,P\varphi_i = -P\varphi_i,$$

and so $\overline{\partial}\,\eta_i = -\varphi_i$ as desired. ∎

Next, we need to give a proof of convergence as in 5.1.3. This is trivial: since P is a linear differential operator of order 0, inequality (5.1.7) still holds (with possibly different constant values) and the remainder of the argument in 5.1.3 is applicable here again.

Finally, we can carry the arguments in 5.1.4 verbatim, and this concludes the proof.

Remark 6.3.1. The Tian-Todorov theorem is actually still valid for complex manifolds which admit non-vanishing holomorphic $(n,0)$-forms and which merely obey the $\partial\bar{\partial}$-lemma as opposed to having the stronger property of admitting Kähler metrics. See for instance (Popovici, 2013, p. 4).

CHAPTER VII

KODAIRA-SPENCER DEFORMATION THEORY

In this chapter we present an alternative framework for the deformation theory of compact complex manifolds, mainly developed by Kodaira and Spencer in (Kodaira *et al.*, 1958), (Kodaira et Spencer, 1958a), (Kodaira et Spencer, 1958b) and (Kodaira et Spencer, 1960), hence the title of this chapter. Two standard references are (Morrow, 2006) and (Kodaira, 2005).

In their approach, Kodaira and Spencer begin with so-called smooth differentiable families instead of one-parameter deformations as we did. One advantage is that, in this context, Kodaira-Spencer classes may be explicitly computed. For expediency, we will make use of the framework developed in chapter 4; however, the reader should be aware that the framework of Kodaira and Spencer can be worked out entirely independently of ours.

7.1 Smooth differentiable families

The following definition is the starting point of Kodaira-Spencer deformation theory:

Definition 7.1.1. A *smooth differentiable family* (SDF) is a triple $(\mathcal{M}, I, \bar{\omega})$ where $\mathcal{M}$ is a smooth manifold of dimension $2n + 1$, $I \subseteq \mathbf{R}$ an open interval around $0 \in \mathbf{R}$ and $\bar{\omega} : \mathcal{M} \to I$ a smooth map such that the following holds:

(i) $\bar{\omega}$ is surjective;

(ii) at every point of $\mathcal{M}$, the Jacobian of $\bar{\omega}$ has rank 1 (thus every $t \in I$ is regular);

(iii) for every $t \in I$, the submanifold $M_t = \bar{\omega}^{-1}(\{t\}) \subseteq \mathcal{M}$ is compact and connected;

(iv) there is a countable, locally finite cover by open sets $\{\mathcal{U}_i\}$ of $\mathcal{M}$ and functions $z_i : \mathcal{U}_i \to \mathbf{C}^n$ such that, for every $t \in I$, $\{(\mathcal{U}_i \cap M_t, z_i)\}$ is a holomorphic atlas for M_t.

Example 7.1.1. Let B be the upper complex plane. To each $z \in B$ we associate a lattice $L_z = \{mz + n \ : \ m, n \in \mathbf{Z}\}$. The quotient $T_z = \mathbf{C}/L_z$ is a complex torus. Let G be the group consisting of the transformations on $B \times \mathbf{C}$ of the form $(z, \zeta) \mapsto (z, \zeta + mz + n)$, where $m, n \in \mathbf{Z}$. Then G acts freely, properly discontinuously and by biholomorphisms on $B \times \mathbf{C}$ and so $\mathcal{N} = B \times \mathbf{C}/G$ is a complex manifold by Proposition 1.2.2. Clearly the projection on the first component $B \times \mathbf{C} \to B$ factors through the quotient map $\pi : B \times \mathbf{C} \to \mathcal{N}$. Let $\eta : \mathcal{M} \to B$ be the map occurring in this factorization; note that η, seen as a map between C^∞ manifolds, is a submersion. Let $\gamma : I \to B$ be a regular smooth curve, where I is an open interval around $0 \in \mathbf{R}$. Then if $\varepsilon > 0$ is sufficiently small, $\gamma : (-\varepsilon, \varepsilon) \to B$ is an embedding. Define $\mathcal{M} = \eta^{-1}(\gamma(-\varepsilon, \varepsilon))$. It is clear then that $(\mathcal{M}, (-\varepsilon, \varepsilon), \gamma^{-1} \circ \eta)$ is an SDF.

For the remainder of this chapter, $(\mathcal{M}, I, \bar{\omega})$ is an SDF and $\mathcal{M}$ has dimension $2n + 1$. Let $\{\mathcal{U}_i\}$ and $z_i : \mathcal{U}_i \to \mathbf{C}^n$ be as in property (iv) in the definition above.

Theorem 7.1.1. *The map $\bar{\omega}$ is proper.*

Proof. We show that around each $t \in I$ there is a bounded open interval $W \Subset I$ such that $\bar{\omega}^{-1}(\overline{W})$ is compact. It is of course sufficient to show this for $t = 0$.

We write points in $z_i \times \bar{\omega}(\mathcal{U}_i) \subseteq \mathbf{C}^n \simeq \mathbf{R}^{2n}$ as $(x_i^1, \ldots, x_i^{2n}, t)$. We have

$$\bar{\omega}(x_i^1, \ldots, x_i^{2n}, t) = t.$$

Let $\{\rho_i\}$ be a smooth partition of unity subordinate to $\{\mathcal{U}_i\}$ and let X be the vector field $\sum_i \rho_i \frac{\partial}{\partial t}$. In $\mathcal{U}_i$ one has

$$X = \sum_{k=1}^{2n} X_i^k \frac{\partial}{\partial x_i^k} + \frac{\partial}{\partial t}.$$

Let $\gamma : (-a, a) \subseteq \mathbf{R} \to \mathcal{M}$ be an integral curve of X starting at $p \in M_0 = \bar{\omega}^{-1}(\{0\})$. We have $\bar{\omega} \circ \gamma(t) = t$. Since M_0 is compact, there is a small open interval $I' \subseteq \mathbf{R}$ centered around 0 on which are defined all integral curves of X starting in M_0. Let θ be the flow of X. Choose $[-\varepsilon, \varepsilon] \subseteq I'$, $\varepsilon > 0$, and put $S = \theta([-\varepsilon, \varepsilon] \times M_0)$. For $t \in [-\varepsilon, \varepsilon]$ and $p \in M_0$ one has $\bar{\omega} \circ \theta(t, p) = t$. Thus $\theta(\{t\} \times M_0) \subseteq M_t$. Conversely, if $p \in M_t = \bar{\omega}^{-1}(\{t\})$ and $\theta(-t, p)$ is defined, $\theta(-t, p) \in M_0$. Since $\theta(\{-t\} \times \theta(\{t\} \times M_0)) = M_0$, that M_t is connected, and that $\theta(\{t\} \times M_0)$ is non-empty and closed in M_t, this forces $M_t = \theta(\{t\} \times M_0)$. Indeed, otherwise we could find a $p \in M_t$ not in $\theta(\{t\} \times M_0)$ but for which $\theta(-t, p)$ is defined. But this would imply that there is a $q \in \theta(\{t\} \times M_0)$ with $\theta(-t, p) = \theta(-t, q)$, which is impossible. Thus

$$S = \bar{\omega}^{-1}([-\varepsilon, \varepsilon]).$$

The set S is obviously compact, and so we are done. ∎

Ehresmann's fibration theorem ((Dundas, 2013), Theorem 9.5.6) gives us the following important corollary:

Corollary 7.1.1. *Around each $t \in I$ there is an open interval $W \subseteq I$ and a diffeomorphism $\Psi : \bar{\omega}^{-1}(W) \to M_t \times W$ such that the diagram*

$$\begin{array}{ccc} \bar{\omega}^{-1}(W) & \xrightarrow{\Psi} & M_t \times W \\ & {\scriptstyle \bar{\omega}} \searrow \quad \swarrow {\scriptstyle P} & \\ & W & \end{array},$$

commutes, where P is the projection onto the second factor. Furthermore, Ψ is such that $\Psi(p) = (p, t)$ for $p \in M_t$.

Example 7.1.1 is anticipatory of the next corollary. As is well-known, any two complex tori are diffeomorphic though not necessarily biholomorphic. A more general result holds:

Corollary 7.1.2. *For any $t_0, t_1 \in I$, M_{t_0} and M_{t_1} are diffeomorphic.*

Proof. The previous corollary makes it clear that given $t \in I$, $M_{t'}$ is diffeomorphic to M_t for any t' nearby. Since the closed interval bounded by t_0 and t_1 is compact and connected, the result is immediate. ∎

Let us apply Corollary 7.1.1 to $t = 0$. Let $W \subseteq I$ and Ψ be just as in the statement of the corollary. From now on, we write M instead of M_0; thus Ψ is a diffeomorphism from $\bar{\omega}^{-1}(W)$ to $M \times W$. Note that Ψ sends M_t to $M \times \{t\}$. By property (iv) in Definition 7.1.1, $\{(\Psi(\mathcal{U}_i) \cap (M \times \{t\}), z_i \circ \Psi^{-1})\}$ is a holomorphic atlas for $M \times \{t\} \simeq M$. Thus, there is an associated ACS on M, say J_t. Let $\mathcal{J}(M)$ be the space of ACS's on M. The map $\gamma : W \to \mathcal{J}(M)$ which sends t to J_t is a one-parameter deformation of J_0. Accordingly, associated to γ are an infinitesimal deformation and a Kodaira-Spencer class. These do not actually depend on the choice of W and Ψ. Indeed, suppose W' and Ψ' are a another such interval-diffeomorphism pair. Let γ' be the new one-parameter deformation. Then $\gamma'|_{W \cap W'} = \sigma \cdot \gamma|_{W \cap W'}$, where σ is a curve in $\mathrm{Diff}(M)$, the group of diffeomorphisms from M to itself, with $\sigma(0) = \mathrm{Id}$. By our definitions, the infinitesimal deformations, and hence Kodaira-Spencer classes, associated to γ and γ' coincide. Therefore, one can speak unambiguously about the Kodaira-Spencer class of the SDF $(\mathcal{M}, I, \bar{\omega})$. We will see in the next section that it can be computed explicitly.

Remark 7.1.1 (A note on integrability). In the context of Kodaira-Spencer deformation theory, $\theta \in H^1(M, \Theta T^{1,0}M)$ is integrable if there is an SDF whose Kodaira-Spencer class is θ. Proving the corresponding versions of the theorem of existence and the Tian-Todorov theorem is easy from our work in 5.1.4. Indeed, as in 5.1.4, let θ be a fixed element of $H^1(M, \Theta T^{1,0}M)$. We had considered the complex manifold $M \times V_\varepsilon$, where V_ε is the open disc in $\mathbf{C}$ with radius $\varepsilon > 0$. Let $\mathrm{pr}_2 : M \times V_\varepsilon \to V_\varepsilon$ be the projection onto the second factor. It is easy to see that $(\mathrm{pr}_2^{-1}((-\varepsilon, \varepsilon)), (-\varepsilon, \varepsilon), \mathrm{pr}_2)$ is an SDF whose Kodaira-Spencer class is θ.

7.2 Computation of Kodaira-Spencer classes

We are now going to compute the Kodaira-Spencer class of the SDF $(\mathcal{M}, I, \bar{\omega})$. Unless otherwise stated, we retain the notation from the previous section. We regard M as a complex manifold by equipping it with the holomorphic atlas $\{(\mathcal{U}_i \cap M, z_i)\}$. By identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ via $(z^1, \ldots, z^n) \mapsto (\Re(z^1), \ldots, \Re(z^n), \Im(z)^1, \ldots, \Im(z)^n)$, we see that $z_i \times \bar{\omega} : \mathcal{U}_i \to \mathbf{C}^n \times \mathbf{R}$ is a C^∞ coordinate chart of $\mathcal{M}$. Consider the transition function $(z_j \times \bar{\omega}) \circ (z_i \times \bar{\omega})^{-1}$. It takes the form

$$(z_i^1, \ldots, z_i^n, t) \mapsto (z_j^1, \ldots, z_j^n, t) = (f_{ji}(z_i^1, \ldots, z_i^n, t), t),$$

where f_{ji} is a $\mathbf{C}^n$-valued function, holomorphic in $z_i^1, \ldots, z_i^n$ and C^∞ in t. Note that on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$,

$$f_{ik}(z_i^1, \ldots, z_i^n, t) = f_{ij}(f_{jk}(z_k^1, \ldots, z_k^n, t), t). \tag{7.2.1}$$

Let $\Theta T^{1,0}M$ be the sheaf of germs of holomorphic vector fields over M. The Kodaira-Spencer class in question, which we denote by θ, is an element in $H^1(M, \Theta T^{1,0}M)$. It is easy to show that we can suppose that $(z_i \times \bar{\omega})(\mathcal{U}_i)$ has the form $U_i \times I_i$ where U_i is a polydisc in $\mathbf{C}^n$ and I_i is an open subinterval of I. Thus we may write

$$\mathcal{M} \simeq \bigsqcup_i U_i \times I_i / \sim, \tag{7.2.2}$$

where $\sim$ indicates we make the identifications $(z_j, t) = (f_{jk}(z_k, t), t)$ for every $1 \leq j, k \leq n$.

Now since M is compact and $\{\mathcal{U}_i\}$ is locally finite, there are finitely many i's such that $M \cap \mathcal{U}_i \neq \emptyset$. We can suppose that these are $i = 1, \dots, N$, so through (7.2.2) we have

$$M \simeq \bigsqcup_{i=1}^{N} U_i \times \{0\}/\sim . \tag{7.2.3}$$

Now let $V \Subset I_1 \cap \cdots \cap I_N$. By Theorem 7.1.1, $\bar{\omega}^{-1}(\overline{V})$ is a compact subset of $\mathcal{M}$. Because $\bar{\omega}$ is a submersion and in particular an open map, then by shrinking V if need be we can arrange that, through (7.2.2),

$$\bar{\omega}^{-1}(V) \simeq \bigsqcup_{i=1}^{N} U_i \times V/\sim . \tag{7.2.4}$$

Thus

$$M_t \simeq \bigsqcup_{i=1}^{N} U_i \times \{t\}/\sim \tag{7.2.5}$$

for $t \in V$. However, the identifications on the RHS's of (7.2.5) and (7.2.3) are not necessarily the same. Indeed they are respectively of the form $(z_j, 0) = (f_{jk}(z_k, 0), 0)$ and $(z_j, t) = (f_{jk}(z_k, t), t)$, but the behaviours of $f_{jk}(\cdot, 0)$ and $f_{jk}(\cdot, t)$ need not agree. So for t close to 0, M_t and M_0 are obtained by gluing the same polydiscs but in *possibly different ways*, as is dictated by the change of the functions f_{jk} in the t variable. This is the key observation that Kodaira and Spencer make in the development of their deformation theory. As we will see, the Kodaira-Spencer class θ is very much a measure of the variation of the functions f_{jk} in the t variable.

We define the following vector field on $\mathcal{U}_j \cap \mathcal{U}_k \subseteq \mathcal{M}$, $1 \leq j, k \leq n$:

$$\theta_{jk}(z_k, t) = \sum_{\mu=1}^{n} \frac{\partial f_{jk}^{\mu}}{\partial t}(z_j, t) \frac{\partial}{\partial z_j^{\mu}}. \tag{7.2.6}$$

Clearly θ_{jk} is a holomorphic vector field on $\mathcal{U}_j \cap \mathcal{U}_k \cap M$. Differentiating (7.2.1) with respect to t and putting $t = 0$, we find the identity

$$\theta_{jk}|_M = \theta_{jh}|_M + \theta_{hk}|_M$$

on $M \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_h$. This implies $\theta_{jk}\big|_M + \theta_{kh}|_M + \theta_{hj}\big|_M = 0$ and $\theta_{jk}\big|_M = -\,\theta_{kj}\big|_M$. Thus $\{\theta_{jk}\big|_M\}$ forms a 1-cochain with respect to the cover $\{\mathcal{U}_i \cap M \;:\; 1 \le i \le N\}$ of M. Since this cover consists of polydiscs, the cohomology groups with respect to it coincide with the Čech cohomology groups of $\Theta T^{1,0}M$.[1] Thus $\{\theta_{jk}\big|_M\}$ defines a cohomology class $[\{\theta_{jk}\big|_M\}] \in H^1(M, \Theta T^{1,0}M)$. This cohomology class turns out to be none other than θ. To prove this, we first need to say a bit more about the proof of Proposition 4.2.4 (we use here the same notation as Section 4.2). Specifically, we look at the isomorphism in question when $p = 1$.

Let $\mathscr{A}^{p,q}T^{1,0}M$ be the sheaf of germs of C^∞ $T^{1,0}M$-valued (p,q)-forms. Quite clearly, $\mathscr{A}^{p,q}T^{1,0}M$ is fine. Let $\mathscr{Z}^{p,q}T^{1,0}M = \{u \in \mathscr{A}^{p,q}T^{1,0}M \;:\; \bar\partial u = 0\}$. By the Dolbeaut lemma (Lemma 1.3.1), we have the following short exact sequence:

$$0 \to \Theta T^{1,0}M \to \mathscr{A}^{0,0}T^{1,0}M \overset{\bar\partial}{\to} \mathscr{Z}^{0,1}T^{1,0}M \to 0. \tag{7.2.7}$$

The cohomology groups $H^0(M, \mathscr{A}^{0,0}T^{1,0}M)$ and $H^0(M, \mathscr{Z}^{0,1}T^{1,0}M)$ are naturally identified with $\Gamma(M, \mathscr{A}^{0,0}T^{1,0}M)$ and $\ker\bar\partial \subseteq \Gamma(M, \mathscr{A}^{0,1}T^{1,0}M)$ respectively, and the following diagram commutes:

$$\begin{array}{ccc}
H^0(M, \mathscr{A}^{0,0}T^{1,0}M) & \xrightarrow{\;\bar\partial\;} & H^0(M, \bar\partial\mathscr{A}^{0,1}T^{1,0}M) \\
\simeq\big\downarrow & & \big\downarrow\simeq \\
\Gamma(M, \mathscr{A}^{0,0}T^{1,0}M) & \xrightarrow[\bar\partial]{\qquad\qquad} & \ker\bar\partial
\end{array}$$

.

Together with the fact that $\mathscr{A}^{0,0}T^{1,0}M$ is fine, this means that the induced long exact sequence in cohomology looks like so:

[1]See Lemma 5.1 and Lemma 5.2 in (Kodaira, 2005).

$$0 \to H^0(M, \Theta T^{1,0}M) \to \Gamma(M, \mathscr{A}^{0,0}T^{1,0}M) \xrightarrow{\bar\partial} \ker\bar\partial \xrightarrow{\delta^*} \qquad (7.2.8)$$
$$H^1(M, \Theta T^{1,0}M) \to 0 \to \ldots,$$

where δ^* is the transition morphism. Thus δ^* descends to an isomorphism from $\ker\bar\partial/\bar\partial\,\Gamma(M, \mathscr{A}^{0,0}T^{1,0}M) \simeq \ker\bar\partial_1/\mathrm{im}\,\bar\partial_0$ to $H^1(M, \Theta T^{1,0}M)$. This isomorphism is precisely the one appearing in Proposition 4.2.4.

Let W and Ψ be as in the previous section. We identify $\bar\omega^{-1}(W)$ with $M \times W$ with via Ψ and thus designate an arbitrary point of $\bar\omega^{-1}(W)$ by (z,t). Note that $\bar\omega(z,t) = t$.

We parametrize $\mathcal{J}(M)$ around J_0 by $\mathcal{A}^1T^{1,0}M$. Let $\lambda : W \to \mathcal{A}^1T^{1,0}M$ be the curve corresponding to $t \mapsto J_t$ through this parametrization; note that $\lambda(0) = 0$. Then

$$\theta = \lambda'(0) + \bar\partial\mathcal{A}^0T^{1,0}M,$$

From Remark 3.2.1, the $\bar\partial$-operator on $C^\infty_{\mathbb{C}}(M)$ associated to the integrable ACS J_t, which we denote by $\bar\partial_t$, is given by

$$\bar\partial_t = \bar\partial_0 + \lambda(t), \qquad (7.2.9)$$

where $\lambda(t)$ is an interpreted as an operator taking $C^\infty_{\mathbb{C}}(M)$ to $C^\infty(M, T^{0,1}M)$ in the obvious way. Let $\mathrm{pr}_M : M \times W \to M$ be the projection onto the first factor. It is clear that we can define a first order differential operator

$$\bar\partial : C^\infty_{\mathbb{C}}(M \times W) \to C^\infty(M \times W, \mathrm{pr}_M^* T^{1,0}M)$$

so that the following diagram commutes:

$$\begin{array}{ccc} C^\infty_{\mathbb{C}}(M) & \xrightarrow{\ \bar\partial\ } & C^\infty(M, T^{0,1}M) \\ \circ\,\mathrm{pr}_M \downarrow & & \downarrow \circ\,\mathrm{pr}_M \\ C^\infty_{\mathbb{C}}(M \times W) & \xrightarrow[\bar\partial]{} & C^\infty(M \times W, \mathrm{pr}_M^* T^{0,1}M) \end{array}$$

.

We also define a section of $\mathrm{pr}_M^* \bigwedge^{0,1} T^*M \otimes_{\mathbf{C}} T^{1,0}M$, which we denote by $\hat{\lambda}$, by putting

$$\hat{\lambda}(z,t) = \lambda(t)(z).$$

Finally we can define another first order differential operator

$$\hat{\bar{\partial}} : C^\infty_{\mathbf{C}}(M \times W) \to C^\infty(M \times W, \mathrm{pr}_M^* T^{0,1}M)$$

such that the following diagram commutes for all $t \in W$:

$$\begin{array}{ccc} C^\infty_{\mathbf{C}}(M \times W) & \xrightarrow{\hat{\bar{\partial}}} & C^\infty(M \times W, \mathrm{pr}_M^* T^{0,1}M) \\ \circ\, \iota_t \downarrow & & \downarrow \circ\, \iota_t \\ C^\infty_{\mathbf{C}}(M) & \xrightarrow[\bar{\partial}_t]{} & C^\infty(M, T^{0,1}M) \end{array},$$

where $\iota_t : M \to M \times W$ is the map $z \mapsto (z,t)$. Because of (7.2.9), we have

$$\hat{\bar{\partial}} = \bar{\partial} + \hat{\lambda}, \tag{7.2.10}$$

where $\hat{\lambda}$ is interpreted as an operator taking $C^\infty_{\mathbf{C}}(M \times W)$ to $C^\infty(M \times W, \mathrm{pr}_M^* T^{0,1}M)$. We can now prove

Proposition 7.2.1. *The cohomology class* $[\{\theta_{jk}|_M\}]$ *is exactly* θ.

Proof. Since $z_i^\alpha(z,t) = f_{ik}^\alpha(z_k(z,t),t)$,

$$\begin{aligned} \frac{\partial z_i^\alpha}{\partial t}(z,t) &= \sum_{\beta=1}^n \frac{\partial f_{ik}^\alpha}{\partial z_k^\beta}(z_k(z,t),t)\frac{\partial z_k^\beta}{\partial t}(z,t) + \frac{\partial f_{ik}^\alpha}{\partial t}(z_k(z,t),t) \\ &= \sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_k^\beta}(z,t)\frac{\partial z_k^\beta}{\partial t}(z,t) + \frac{\partial f_{ik}^\alpha}{\partial t}(z_k(z,t),t), \end{aligned}$$

for $(z,t) \in \mathcal{U}_i \cap \bar{\omega}^{-1}(W)$. Consider the following vector field defined on $\mathcal{U}_i \cap \bar{\omega}^{-1}(W)$:

$$\xi_i = \sum_{\alpha=1}^n \frac{\partial z_i^\alpha}{\partial t}\frac{\partial}{\partial z_i^\alpha}.$$

On $\mathcal{U}_i \cap \mathcal{U}_k \cap \bar{\omega}^{-1}(W)$,

$$\begin{aligned}
\xi_i(z,t) - \xi_k(z,t) &= \sum_{\alpha=1}^{n}\left(\sum_{\beta=1}^{n}\frac{\partial z_i^\alpha}{\partial z_k^\beta}(z,t)\frac{\partial z_k^\beta}{\partial t}(z,t) + \frac{\partial f_{ik}^\alpha}{\partial t}(z_k(z,t),t)\right)\frac{\partial}{\partial z_i^\alpha} - \\
&\quad \sum_{\beta=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial z_i^\alpha}{\partial z_k^\beta}(z,t)\frac{\partial z_k^\beta}{\partial t}(z,t)\frac{\partial}{\partial z_i^\alpha} \\
&= \theta_{ik}(z,t).
\end{aligned}$$

Since $\bar{\partial}_t\, z_i^\alpha\big|_{M_t} = 0$, we have

$$\hat{\bar{\partial}} z_i^\alpha = 0.$$

Thus

$$\bar{\partial}\,\frac{\partial z_i^\alpha}{\partial t} = \mathcal{L}_{\partial/\partial t}\,\bar{\partial}\, z_i^\alpha = \mathcal{L}_{\partial/\partial t}(-\hat{\lambda}\cdot z_i^\alpha),$$

implying that $\bar{\partial}\,\frac{\partial z_i^\alpha}{\partial t}\Big|_{t=0} = -\lambda'(0)\cdot z_i^\alpha$ on $\mathcal{U}_i \cap M$. In particular,

$$\begin{aligned}
\bar{\partial}\,\xi_i|_M &= \sum_{\alpha=1}^{n}\bar{\partial}\,\frac{z_i^\alpha}{\partial t}\Big|_{t=0}\otimes\frac{\partial}{\partial z_i^\alpha} \\
&= \sum_{\alpha=1}^{n}-\lambda'(0)\cdot z_i^\alpha\otimes\frac{\partial}{\partial z_i^\alpha} \\
&= -\lambda'(0).
\end{aligned}$$

Seeing $\{\xi_i|_M\}$ and $\lambda'(0)$ as 0-cochains of the Čech complexes associated to $\mathscr{A}^{0,0}T^{1,0}M$ and $\mathscr{A}^{0,1}T^{1,0}M$ respectively with respect to the cover $\{\mathcal{U}_i \cap M\}$, one has $\bar{\partial}\{\xi_i|_M\} = \lambda'(0)$. Clearly then $\lambda'(0) \in \ker\bar{\partial} = \Gamma(M, \bar{\partial}\mathscr{A}^{0,0}T^{1,0}M)$.

Let δ be the Čech differential associated to the Čech complex of $\Gamma(M, \mathscr{A}^{0,0}T^{1,0}M)$. The above shows that $\delta\{\xi_i|_M\} = -\{\theta_{jk}|_M\}$. In view of (7.2.7), (7.2.8) and the definition of δ^*, this means

$$\delta^*([\lambda'(0)]) = \theta,$$

and the claim follows. ∎

The above effectively gives us a way to explicitly compute the Kodaira-Spencer class of an SDF, which completes what we had set out to do.

APPENDIX A

THE NEWLANDER-NIRENBERG THEOREM

In Chapter 1, we gave two definitions of complex manifolds, namely definitions 1.2.1 and 1.2.2. We saw that Definition 1.2.2 is subsumed by Definition 1.2.1. In this appendix, we prove the converse.

Let J be an ACS on a C^∞ manifold M of dimension $2n$. The anti-holomorphic tangent bundle $T^{0,1}M$ is a rank n subbundle of $T^{\mathbf{C}}M$. Recall that J is said to be formally integrable if, among other characterizations, $T^{0,1}M$ is involutive, i.e. for every vector fields Z_1 and Z_2 of bidegree $(0, 1)$, the vector field $[Z_1, Z_2]$ is also of bidegree $(0, 1)$. On the other hand, we say that J is integrable if it is induced by a complex manifold (as per Definition 1.2.2) of which M is the underlying C^∞ manifold. What must effectively be shown is that if J is formally integrable, then it is also integrable (the converse being already known). This is precisely the Newlander-Nirenberg Theorem.

Theorem A.0.1 (Newlander-Nirenberg). *Let J be an ACS on a C^∞ manifold M of dimension $2n$. If J is formally integrable, then J is integrable.*

Before discussing the proof of the theorem, we need to state two well-known results from the theory of elliptic systems on which the proof relies; we do not prove them. Let $\Omega \subseteq \mathbf{R}^k$ be open and suppose we are given K real-analytic functions

$$F_i : \Omega \times \mathbf{R}^{kN} \times \mathbf{R}^{k^2 N} \times \cdots \times \mathbf{R}^{k^m N} \to \mathbf{R}, \ \ i = 1, \ldots, K.$$

Consider the analytic system of equations

$$F_i(x, u, Du, \ldots, D^m u) = 0, \;\; i = 1, \ldots, K \tag{A.0.1}$$

of order m with K equations for $N \le K$ unknown functions $u = (u_1, \ldots, u_N)$ and x varying in Ω. We say that the system is *elliptic at* $u \in C^m(\Omega, \mathbf{R}^N)$ if the linearization of $F = (F_1, \ldots, F_K)$ at u is an elliptic linear differential operator.

Theorem A.0.2 (Regularity of solutions of (over)determined systems, (Besse, 1987, p. 467)). *If (A.0.1) is elliptic at $u \in C^m(\Omega, \mathbf{R}^N)$ and u is a solution, then u is real-analytic in Ω.* ■

Theorem A.0.3 (Local solvability, (Besse, 1987, p. 469)). *Suppose now that $N = K$, that (A.0.1) is elliptic at $u_0 \in C^m(\Omega, \mathbf{R}^N)$ and that u_0 satisfies (A.0.1) at one point $x_0 \in \Omega$ (u_0 is called an infinitesimal solution at x_0). Then there are $\varepsilon_0 > 0$, $C > 0$, $1 > \sigma > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there is a $u_\varepsilon \in C^m(B_\varepsilon(0), \mathbf{R}^N)$ solving (A.0.1) in $B_\varepsilon(0)$ and satisfying the following bounds:*

$$|D^\alpha u_\varepsilon(x) - D^\alpha u_0(x)| \le C\varepsilon^{m-|\alpha|+\sigma}, \;\; 0 \le |\alpha| \le m.$$

■

We now return to the Newlander-Nirenberg theorem. The following lemma says it is really a local result.

Lemma A.0.1 ((Kobayashi et Nomizu, 1963, Appendix 8, p. 321)). *If around every point of M there is a neighbourhood $U \subseteq M$ and C^∞ $\mathbf{C}$-valued functions $f^1, \ldots, f^n$ such that the differential forms $df^1, \ldots, df^n$ span $\bigwedge^{1,0} T^*M$ in U, then J is integrable. Conversely, if J is integrable then such neighbourhoods and functions exist around every point of M.* ■

Thus to prove the Newlander-Nirenberg theorem, one can assume that M is a domain in $\mathbf{R}^{2n} \simeq \mathbf{C}^n$ around the origin and show that there is a neighbourhood $U \subseteq M$ of 0 and functions $f^1, \ldots, f^n$ as in the lemma above. Equivalently, we would like to show that in some neighbourhood of 0 there are vector fields $L_1, \ldots, L_n$ spanning $T^{0,1}M$ and n linearly independent solutions $f^1, \ldots, f^n$ of the system

$$L_i f = 0, \quad i = 1, \ldots, n.$$

In the case $T^{0,1}M$ is analytic, meaning that $T^{0,1}M$ admits an analytic local frame[1] around every point of M, this is relatively simple to establish.[2] We will take this as given and prove the theorem by showing that the general case can be reduced to the analytic one.[3]

We identify $\mathbf{R}^2 n$ with $\mathbf{C}^n$ via $(x^1, \ldots, x^n, y^1, \ldots, y^n) \mapsto (z^1, \ldots, z^n)$ where $z^j = x^j + \sqrt{-1}y^j$. Choose a basis $\{s_1, \ldots, s_n\} \subseteq \bigwedge^{1,0} T_0^{*\mathbf{C}}M$ for $\bigwedge^{1,0} T_0^{*\mathbf{C}}M$. Without loss of generality, we may assume that $dz^j(0) = s_j$, $j = 1, \ldots, n$. Now let $\omega_1, \ldots, \omega_n$ be 1-forms spanning $\bigwedge^{1,0} T^{*\mathbf{C}}M$ in a neighbourhood of 0 with $\omega_j(0) = dz^j(0) = s_j$. Suppose L be a complex vector field defined in a neighbourhood of 0. Writing

$$L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z^j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial \overline{z}^j},$$

we see that, if L is of bidegree (0, 1), the coefficient functions a_k must vanish at 0. Let L_k, $k = 1, \ldots, n$, be complex vector fields of bidegree (0, 1) that span $T^{0,1}M$ in a

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[1] A $\mathbf{C}$-valued function f defined on an open subset of M is said to be analytic if around each point of its domain there is a chart $\varphi : U \to \mathbf{R}^{2n}$ such that $f \circ \varphi^{-1}$ is real-analytic. A (local) complex vector field X is said to be analytic if Xf is analytic whenever f is.

[2] See (Kobayashi et Nomizu, 1969, Appendix 8) for a proof.

[3] This proof is due to Malgrange ((Malgrange, 1969)). It can be found in essentially the same form as presented here in (Nirenberg, 1973), (Karzan, 1993) and (Berhanu *et al.*, 2008).

neighbourhood of 0. For notational convenience, we put

$$\frac{\partial}{\partial z} = \left[\frac{\partial}{\partial z^1} \cdots \frac{\partial}{\partial z^n}\right]^T \tag{A.0.2}$$

$$\frac{\partial}{\partial \overline{z}} = \left[\frac{\partial}{\partial \overline{z}^1} \cdots \frac{\partial}{\partial \overline{z}^n}\right]^T \tag{A.0.3}$$

$$\mathbf{L} = [L_1 \cdots L_n]^T .$$

Writing

$$\mathbf{L} = A\frac{\partial}{\partial z} + B\frac{\partial}{\partial \overline{z}}$$

where A and B are $n \times n$ matrix functions, it must be that $A(0) = 0$ and that B is nonsingular at (and hence in a neighbourhood of) 0. We substitute $B^{-1}\mathbf{L}$ for $\mathbf{L}$ so that

$$\mathbf{L} = \frac{\partial}{\partial \overline{z}} + C\frac{\partial}{\partial z},$$

with $C(0) = 0$. For reasons that will come to light in the sequel, we will assume that $C(z) = O(|z|^2)$. The following lemma shows that this is justified.

Lemma A.0.2. *By a change of coordinates of the form $z \mapsto z + Q(z, \overline{z})$ where Q is a homogeneous quadratic polynomial, we can (after possibly changing $\mathbf{L}$) suppose that $C(z) = O(|z|^2)$.*

Proof. We carry the proof only for $n = 1$, generalizing to higher dimensions is straightforward. The initial data is a vector field around 0

$$L(z) = \frac{\partial}{\partial \overline{z}}(z) + c(z) \cdot \frac{\partial}{\partial z}(z),$$

where c is a C^∞ function with $c(0) = 0$. We define $Q(z, \overline{z}) = -\frac{\partial c}{\partial z}(0) \cdot z\overline{z} - \frac{1}{2}\frac{\partial c}{\partial \overline{z}}(0) \cdot \overline{z}\,\overline{z}$. The map $z \mapsto \tilde{z} = z + Q(z, \overline{z})$ defines a diffeomorphism close to 0 and hence a change of complex coordinates. One has

$$\begin{aligned} L(\tilde{z}) \quad = \quad & \left(\frac{\partial \overline{\tilde{z}}}{\partial \overline{z}}(z(\tilde{z})) + c(z(\tilde{z})) \cdot \frac{\partial \overline{\tilde{z}}}{\partial z}(z(\tilde{z}))\right) \frac{\partial}{\partial \overline{\tilde{z}}}(\tilde{z}) + \\ & \left(\frac{\partial \tilde{z}}{\partial \overline{z}}(z(\tilde{z})) + c(z(\tilde{z})) \cdot \frac{\partial \tilde{z}}{\partial z}(z(\tilde{z}))\right) \frac{\partial}{\partial \tilde{z}}(\tilde{z}). \end{aligned}$$

Now,

$$\begin{aligned}\frac{\partial \tilde{z}}{\partial \overline{z}}(z) + c(z)\cdot\frac{\partial \tilde{z}}{\partial z}(z) &= -\frac{\partial c}{\partial z}(0)\cdot z - \frac{\partial c}{\partial \overline{z}}(0)\cdot \overline{z} + \\ &\quad \left(\frac{\partial c}{\partial z}(0)\cdot z + \frac{\partial c}{\partial \overline{z}}(0)\cdot \overline{z} + O(|z|^2)\right)(1+O(|z|)) \\ &= O(|z|^2).\end{aligned}$$

At the same time, $|\tilde{z}(z)|^2 = \Theta(|z|^2)$. It follows that

$$\frac{\partial \tilde{z}}{\partial \overline{z}}(z(\tilde{z})) + c(z(\tilde{z}))\cdot\frac{\partial \tilde{z}}{\partial z}(z(\tilde{z})) = O(|\tilde{z}|^2).$$

In a small neighbourhood of 0,

$$\left(\frac{\partial \overline{\tilde{z}}}{\partial \overline{z}}(z(\tilde{z})) + c(z(\tilde{z}))\cdot\frac{\partial \overline{\tilde{z}}}{\partial z}(z(\tilde{z}))\right)^{-1} L(\tilde{z})$$

is defined and we see from the claim is proven if we substitute the above for L. ■

Suppose H is a diffeomorphism defined near $0 \in \mathbf{R}^{2n} \simeq \mathbf{C}^n$ such that $\left[\frac{\partial H}{\partial z}\right]$ is invertible at 0. We put $w = H(z)$ and define $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial w}}$ and $\frac{\boldsymbol{\partial}}{\boldsymbol{\partial \overline{w}}}$ similarly as in (A.0.2) and (A.0.3). We have

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}} = \left[\frac{\partial H}{\partial z}\right]^T \frac{\boldsymbol{\partial}}{\boldsymbol{\partial w}} + \left[\frac{\partial \overline{H}}{\partial z}\right]^T \frac{\boldsymbol{\partial}}{\boldsymbol{\partial \overline{w}}},\quad \frac{\boldsymbol{\partial}}{\boldsymbol{\partial \overline{z}}} = \left[\frac{\partial H}{\partial \overline{z}}\right]^T \frac{\boldsymbol{\partial}}{\boldsymbol{\partial w}} + \left[\frac{\partial \overline{H}}{\partial \overline{z}}\right]^T \frac{\boldsymbol{\partial}}{\boldsymbol{\partial \overline{w}}}.$$

We are thus able to obtain from $L_1, \ldots, L_n$ new vector fields $\tilde{L}_1, \ldots, \tilde{L}_n$ spanning $T^{0,1}M$ in some neighbourhood of 0 defined in terms of the new variables $w^1, \ldots, w^n$ by putting

$$\tilde{\mathbf{L}} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial \overline{w}}} + D\frac{\boldsymbol{\partial}}{\boldsymbol{\partial w}}, \tag{A.0.4}$$

where $\tilde{\mathbf{L}} = \left[\tilde{L}_1 \cdots \tilde{L}_n\right]^T$ and D is a $n \times n$ matrix function given by

$$D(w) = \left(\left[\frac{\partial \overline{H}}{\partial \overline{z}}(z)\right]^T + C(z)\left[\frac{\partial \overline{H}}{\partial z}(z)\right]^T\right)^{-1} \tag{A.0.5}$$

$$\left(\left[\frac{\partial H}{\partial \overline{z}}(z)\right]^T + C(z)\left[\frac{\partial H}{\partial z}(z)\right]^T\right)\Bigg|_{z=H^{-1}(w)}.$$

Note that the expression in the left parentheses is indeed invertible close to 0, since $C(0) = 0$ and H is a diffeomorphism. Now since $T^{0,1}M$ is involutive, $[\tilde{L}_j, \tilde{L}_k]$ for $j,k = 1,\dots,n$ is a linear combination of $\tilde{L}_1,\dots,\tilde{L}_n$. When developing $[\tilde{L}_j, \tilde{L}_k]$, we see that terms of the form $\frac{\partial}{\partial \overline{w}^h}$, $h = 1,\dots,n$, do not appear. In view of (A.0.4), this implies that $[\tilde{L}_j, \tilde{L}_k] = 0$ for $j,k = 1,\dots,n$. Writing $D = [d_{jk}]$, this is equivalent to

$$\frac{\partial d_{kl}}{\partial \overline{w}^j} - \frac{\partial d_{jl}}{\partial \overline{w}^k} - \sum_{r=1}^{n} \left(d_{kr} \frac{\partial d_{jl}}{\partial w^r} - d_{jr} \frac{\partial d_{kl}}{\partial \overline{w}^j} \right) = 0, \;\; j,k,l = 1,\dots,n \text{ and } j < k. \quad \text{(A.0.6)}$$

Consider the additional equations

$$\sum_{j=1}^{n} \frac{\partial d_{jk}}{\partial w^j} = 0, \;\; k = 1,\dots,n. \quad \text{(A.0.7)}$$

The totality of the equations in (A.0.6) and (A.0.7) make up a system of differential equations in the unknowns $\Re d_{11},\dots,\Re d_{nn},\Im d_{11},\dots,\Im d_{nn}$ and real variables $\Re w^1,\dots,\Re w^n,\Im w^1,\dots,\Im w^n$. Putting $V = (\Re d_{11},\dots,\Re d_{nn},\Im d_{11},\dots,\Im d_{nn})$, this system has the form

$$T[V] + \Gamma(V, DV) = 0, \quad \text{(A.0.8)}$$

where T is a linear differential operator (with constant coefficients) and Γ a bilinear form. It is easy though a bit tedious to show that the system

$$T[V] = 0$$

is elliptic.[4] It follows that there is a $\delta > 0$ so that if $||D(0)|| < \delta$, (A.0.8) is elliptic at (and hence in a neighbourhood of) 0. If in addition d solves (A.0.8) close to 0, it is the solution of an overdetermined analytic elliptic system and is thus analytic by Theorem A.0.2. In view of (A.0.4), this means that if we can show that there is an H inducing

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[4]Given $\xi = (\xi_1,\dots,\xi_{2n}) \in \mathbf{R}^{2n}$ with $\xi_p \neq 0$ for some $1 \le p \le n$, one shows that in the matrix of the principal symbol σ_ξ (which depends only on ξ as T has constant coefficients) the minor consisting of the rows corresponding to the equations in (A.0.6) with $j = p$ or $k = p$ and the equations in $(A.0.7)$ has full rank.

such a d, we will have succeeded in showing that $T^{0,1}M$ is analytic and the proof will be complete.

Given (A.0.5) and (A.0.7), we are led to try to find H satisfying

$$\sum_{j=1}^{n} \frac{\partial}{\partial w^j} \left[\left(\left[\frac{\partial \overline{H}}{\partial \overline{z}} \right]^T + C \left[\frac{\partial \overline{H}}{\partial \overline{z}} \right]^T \right)^{-1} \left(\left[\frac{\partial H}{\partial \overline{z}} \right]^T + C \left[\frac{\partial H}{\partial \overline{z}} \right]^T \right) \right]_{jk} = 0, \tag{A.0.9}$$

for $k = 1, \ldots, n$. The above can be regarded as a second order differential system consisting of $2n$ equations in the unknowns $\Re H$ and $\Im H$ and real independent variables $x^1, \ldots, x^n, y^1, \ldots, y^n$, where $\frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^n}$ are interpreted via

$$\begin{bmatrix} \frac{\partial}{\partial w} \\ \frac{\partial}{\partial \overline{w}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial w^1} & \cdots & \frac{\partial}{\partial w^n} & \frac{\partial}{\partial \overline{w}^1} & \cdots & \frac{\partial}{\partial \overline{w}^n} \end{bmatrix}^T = \begin{bmatrix} \left[\frac{\partial H}{\partial z}\right]^T & \left[\frac{\partial \overline{H}}{\partial z}\right]^T \\ \left[\frac{\partial H}{\partial \overline{z}}\right]^T & \left[\frac{\partial \overline{H}}{\partial \overline{z}}\right]^T \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \overline{z}} \end{bmatrix}. \tag{A.0.10}$$

However, (A.0.9) is not a differential system in the usual sense since conditions on the complex Jacobian of H are necessary in order that the equations comprising the system are semantically valid. If we write the system in the form

$$\mathscr{F}(z, H(z), DH(z), D^2H(z)) = 0,$$

then $\mathscr{F}$ is defined in $\mathbf{C}^n \times \mathbf{C}^n \times U \times \mathbf{C}^{n^3}$ where U is an open proper subset of $\mathbf{C}^{n^2}$. Choosing $U' \Subset U$, we can replace $\mathscr{F}$ by a C^∞ function $\tilde{\mathscr{F}}$ defined in $\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}^{n^2} \times \mathbf{C}^{n^3}$ and coinciding with $\mathscr{F}$ on $\mathbf{C}^n \times \mathbf{C}^n \times U' \times \mathbf{C}^{n^3}$, provided we demand that $DH(z) \in U'$ for every z in the domain of H. This additional constraint will not pose a problem for the rest of the argument. We will find it convenient to select U' so that $D\,\mathrm{Id} \in U'$, Id being the identity map $z \mapsto z$. Temporarily consider the differential system

$$\tilde{\mathscr{F}}(z, H(z), DH(z), D^2H(z)) = 0. \tag{A.0.11}$$

Let $\tilde{\mathscr{F}}[\cdot]$ be the map taking $C^2(\overline{B_1(0)}, \mathbf{C}^n)$ into $C^0(\overline{B_1(0)}, \mathbf{C}^n)$ defined naturally by $\tilde{\mathscr{F}}$. Let $G \in C^2(\overline{B_1(0)}, \mathbf{C}^n)$ and put

$$H_t(z) = z + tG(z)$$

with $t \in \mathbf{R}$. Let $\frac{\partial}{\partial w}(t)$ and $\frac{\partial}{\partial \overline{w}}(t)$ be the vector fields given as in (A.0.10) with H_t substituted for H and

$$\mathscr{G}(t) = \left(\left[\frac{\partial \overline{H_t}}{\partial \overline{z}} \right]^T + C \left[\frac{\partial \overline{H_t}}{\partial \overline{z}} \right]^T \right)^{-1} \left(\left[\frac{\partial H_t}{\partial \overline{z}} \right]^T + C \left[\frac{\partial H_t}{\partial \overline{z}} \right]^T \right).$$

It is not difficult to see that each one of $\frac{\partial}{\partial w}(t)$, $\frac{\partial}{\partial \overline{w}}(t)$ and $\mathscr{G}$ is a map defined in some open interval of $0 \in \mathbf{R}$ on which it takes values in a Banach space and is smooth therein.

By Taylor's theorem,

$$\frac{\partial}{\partial w}(t) = \frac{\partial}{\partial z} + R_1(t),$$

where $R_1(t) \in O(t)$. Note that since $C(z) \in O(|z|^2)$, $\frac{\partial}{\partial w}(t)C = \frac{\partial}{\partial z}C = 0$ and so $R_1(t)C = 0$. From the definition of $\mathscr{G}$, it is readily seen that its second order Taylor approximation is given by

$$\mathscr{G}(t) = C + t\left(\left[\frac{\partial G}{\partial \overline{z}} \right]^T + C \cdot E \right) + O(t^2)$$

where E is smooth. Using the facts above, we see that the linearization of (A.0.11) at Id identifies with the mapping

$$G \mapsto \left(\sum_{j=1}^{n} \frac{\partial}{\partial z^j} \left[\frac{\partial G}{\partial \overline{z}} \right]^T_{j1}, \ldots, \sum_{j=1}^{n} \frac{\partial}{\partial z^j} \left[\frac{\partial G}{\partial \overline{z}} \right]^T_{jn} \right) =$$

$$\left(\sum_{j=1}^{n} \frac{\partial G_1}{\partial z^j \partial \overline{z}^j}, \ldots, \sum_{j=1}^{n} \frac{\partial G_n}{\partial z^j \partial \overline{z}^j} \right),$$

and hence (A.0.11) is elliptic at Id. Furthermore, it is obvious that Id satisfies (A.0.9) and hence (A.0.11) at $z = 0$. By Theorem (A.0.3), there are $\varepsilon_0 > 0$, $K > 0$ and $1 > \sigma > 0$

such that for any $0 < \varepsilon \leq \varepsilon_0$, there is an $H_\varepsilon \in C^2(B_\varepsilon(0), \mathbf{C}^n)$ solving (A.0.11) in $B_\varepsilon(0)$ with

$$||\mathrm{Id} - H_\varepsilon||_{C^2(B_\varepsilon(0),\mathbf{C}^n)} \leq K\varepsilon^\sigma.$$

Choosing ε small, we can insure that $DH_\varepsilon(0)$ is non-singular and that $DH_\varepsilon(z) \in U'$ for $z \in B_\varepsilon(0)$. Then H_ε is a diffeomorphism close to 0 and it solves not only (A.0.11) but also (A.0.9) in $B_\varepsilon(0)$. Defining D_ε as in (A.0.5) with H_ε substituted for H, we may arrange that $||D_\varepsilon(0)|| < \delta$ by possibly reducing ε. So if ε is sufficiently small, putting $H = H_\varepsilon$, H is a diffeomorphism close to 0 and is as desired. The proof is thus complete.

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