Virtually fibred Montesinos Links

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1   Introduction

William Thurston conjectured over twenty years ago that every compact hyperbolic 3-manifold whose boundary is a possibly empty union of tori is finitely covered by a manifold which fibres over the circle. If true, it provides a significant amount of global information about the topology of such manifolds. The first non-trivial examples supporting the conjecture were obtained by Gabai [Ga1] (1986) and Reid [Re] (1995). In the 1999 paper [AR], Aitchison and Rubinstein found combinatorial conditions on certain polyhedral decompositions of 3-manifolds which guarantee the existence of a finite cover which fibres over the circle. Chris Leininger showed that every manifold obtained by Dehn filling one component of the Whitehead link exterior is finitely covered by a surface bundle [Le] (2002), and more recently Genevieve Walsh verified Thurston's conjecture for the exteriors of spherical Montesinos link exteriors [Wa] (2005), which includes all 2-bridge links. Jack Button [Bu] (2005) has determined many examples of nonfibred virtually fibred 3-manifolds in the Callahan-Hildebrand-Weeks and Hodgson-Weeks censuses, including over 100 closed examples.

Call a link K in a 3-manifold *virtually fibred* if its exterior has a finite cover which fibres over the circle. In this paper we are interested in examining families of generalised Montesinos links (see §2) with this property. To that end, consider a generalised Montesinos link K and let $(W_K, \tilde{K}) \to (S^3, K)$ be the 2-fold branched cover of (S^3, K) determined by the homomorphism $\pi_1(S^3 \setminus K) \to \mathbb{Z}/2$ which sends each meridinal class to the non-zero element of $\mathbb{Z}/2$. Montesinos proved ([Mo1], [Mo2]) that W_K admits a Seifert fibring $f : W_K \to \mathcal{B}_K$

invariant under the covering involution $\tau : W_K \to W_K$. (It will be assumed that W_K is equipped with such a structure throughout the paper.) Further, by a special case of the orbifold theorem, W_K admits a geometric structure in which $\tilde{K}$ is geodesic and τ an isometry. The type of geometric structure supported by a closed, connected Seifert fibred space V with base $\mathcal{B}$ is uniquely determined by the sign of the orbifold Euler characteristic $\chi(\mathcal{B})$ and whether or not the Euler number $e(V)$ is zero (c.f. §4 of [Sc]). The precise result is detailed in Table 1.

	$\chi(\mathcal{B}_K) > 0$	$\chi(\mathcal{B}_K) = 0$	$\chi(\mathcal{B}_K) < 0$
$e(W_K) = 0$	$\mathbb{S}^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e(W_K) \neq 0$	$\mathbb{S}^3$	Nil	$\widetilde{SL_2}$

Table 1: The type of geometry supported by W_K

We shall say that K has *spherical type* if W_K admits an $\mathbb{S}^3$ geometry, has *Nil type* if W_K admits a *Nil* geometry, has $\widetilde{SL_2}$ *type* if W_K admits an $\widetilde{SL_2}$ geometry, etc. When K is a knot with Alexander polynomial Δ_K, we have $|e(W_K)| = \frac{|H_1(W_K)|}{p_1 p_2 \ldots p_n}$ (Corollary 6.2 of [JN]), while $|H_1(W_K)| = |\Delta_K(-1)| \neq 0$ (see Corollary D3, [Rlf]), and so K has either spherical, *Nil*, or $\widetilde{SL_2}$ type.

This geometric viewpoint is extremely useful for analysing when generalised Montesinos links virtually fibre. For instance, Walsh observed [Wa] that if W_K spherical, it is the quotient of S^3 by a finite group of isometries acting freely, and hence, as $\tilde{K}$ is geodesic, there is a branched cover $(S^3, L) \to (S^3, K)$ where L is a link of great circles. The exterior of any component L_1 of L is a solid torus in which $L \setminus L_1$ is braided. Thus the exterior of L fibres over the circle with fibre a planar surface.

Theorem 1.1 (Walsh) *Every generalised Montesinos link having spherical type virtually fibres.* $\diamond$

In this paper we investigate other types of generalised Montesinos links. For instance, there are exactly two such links of type $\mathbb{S}^2 \times \mathbb{R}$ - the trivial link of two components and a connected sum of two Hopf links (see §4.1). The first does not virtually fibre since its exterior is reducible, while the second does since its exterior is homeomorphic to the product of a thrice-punctured disk and S^1.

In order to describe our results, we must introduce some notation.

Generalised Montesinos links are specified by an ordered set of numbers $(-g; e_0; \frac{q_1}{p_1}, \ldots, \frac{q_n}{p_n})$ where $g, e_0, p_i, q_i \in \mathbb{Z}, g \geq 0, p_i > 1$, and $(q_i, p_i) = 1$, ([Mo1], [Mo2], [BZ]). Different sets

may correspond to the same link and each generalised Montesinos link with $n > 0$ has a representative set with $e_0 = 0$. We write $K(-g; e_0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$ to denote the link associated to $(-g; e_0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$. See Figure 1.
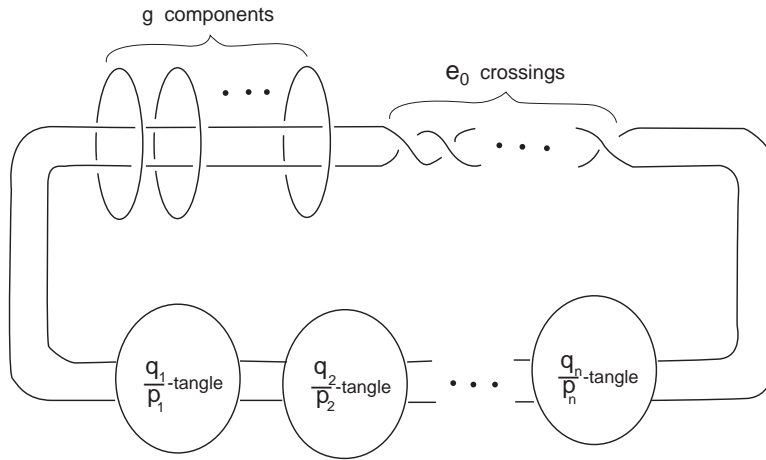

Figure 1: A generalized Montesinos link $K(-g; e_0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$.

A generalised Montesinos link with $g = 0$ is called a *classic Montesinos link*.

Theorem 1.2 *A generalised Montesinos link which is not a classic Montesinos link of type $\widetilde{SL_2}$ virtually fibres if and only if it is not a trivial link of two components. In particular, it virtually fibres if either it is of type $\mathbb{S}^3, \mathbb{E}^3, Nil,$ or $\mathbb{H}^2 \times \mathbb{R}$, or it is a non-classic Montesinos link of type $\widetilde{SL_2}$.*

It follows from the proof of Theorem 1.2 that a stronger statement holds for classic Montesinos links.

Theorem 1.3 *If K is a classic Montesinos link of type $\mathbb{E}^3, Nil,$ or $\mathbb{H}^2 \times \mathbb{R}$, there is a finite degree cover $\Psi : Y \to W_K$ such that Y fibres over the circle in such a way that $L = \Psi^{-1}(\tilde{K})$ is transverse to all the surface fibres.*

This refinement of Theorem 1.2 has an interesting consequence.

Corollary 1.4 *Any finite degree cover of the 3-sphere branched over a classic Montesinos link K of type $\mathbb{E}^3, Nil,$ or $\mathbb{H}^2 \times \mathbb{R}$ which factors through the 2-fold cover $(W_K, \tilde{K}) \to (S^3, K)$ is a closed, virtually fibred 3-manifold.*

In independent work, Jason DeBlois has proved special cases of this corollary using a similar method (see Theorem 1 and §1 of [Dbs]).

Corollary 1.5 *For any $m \geq 1$, the $2m$-fold cyclic cover of S^3 branched over a classic Montesinos link K of type $\mathbb{E}^3$, Nil, or $\mathbb{H}^2 \times \mathbb{R}$ is a closed, virtually fibred 3-manifold. If $m \geq 2$ and K has type Nil or $\mathbb{H}^2 \times \mathbb{R}$, the set of $2m$-fold branched cyclic covers of K is a family of hyperbolic manifolds which represent infinitely many different commensuarbility classes.*

A 3-manifold is *semifibred* if it fibres over a mirrored interval. This is equivalent to the existence of a separating surface which splits the manifold into two twisted I-bundles. Note that while such 3-manifolds do not fibre over the circle in general, they do admit 2-fold covers which do, and therefore provide *trivial* examples of non-fibred, virtually fibred 3-manifolds. Corollary 1.4 can be used to construct infinite families of non-trivial examples. Since the 2^m-fold cyclic cover of S^3 branched over K is a $\mathbb{Z}/2$-homology 3-sphere (see §5 of [Go]), the following corollary provides a simple such family.

Corollary 1.6 *If K is a classic Montesinos knot of Nil type and $m \geq 2$, the 2^m-fold cyclic cover of S^3 branched over K is a closed, hyperbolic, non-fibred, non-semifibred, virtually fibred 3-manifold. Further, these branched covers of K represent infinitely many different commensurability classes.* $\diamondsuit$

This appears to be the first infinite family of closed, non-trivially virtually fibred hyperbolic 3-manifolds known. A particular example which satisfies the hypotheses of Corollary 1.6 is given by the $(3, 3, -3)$-pretzel knot $K = K(0; 0; \frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$ (see Figure 2). In fact, all even order branched cyclic covers of K give non-trivial examples. This follows from the fact that the order of the first homology of the n-fold branched cover of K is the resultant of $\Delta_K(t) = -2t^2 + 5t - 2$ of K and the polynomial $t^n - 1$ (see §5 of [Go]).

The proof of Theorem 1.2 depends on a result of independent interest.

Theorem 1.7 *Let $p : M \to S^1$ be a fibring of a compact orientable 3-manifold, and let L be a link lying in the interior of a finite union of surface fibres such that each component of L is non-trivial in $H_1(M; \mathbb{Q})$. Then M fibres over S^1 with fibres transverse to L. In particular, if M is closed there is a fibring $M \setminus L \to S^1$ such that the boundary of each surface fibre consists of meridians of the components of L.*
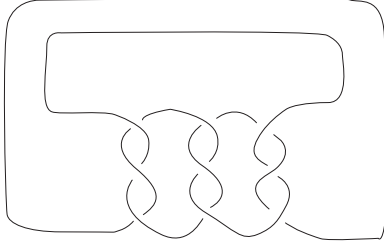
Here are two consequences.

Figure 2: The knot $K(0; 0; \frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$.

Corollary 1.8 *If W is a compact, connected, irreducible, orientable Seifert manifold, F a not necessarily connected essential surface in W, and $\emptyset \neq L \subset int(F)$ a link for which each component is homotopically essential in F, then $W \setminus L$ virtually fibres.*

Corollary 1.9 *Let $p : M \to S^1$ be a fibring of a compact orientable 3-manifold, and let $L \subset M$ be a link lying in the interior of a finite union of surface fibres such that each component of L is non-trivial in $H_1(M; \mathbb{Q})$. Then any finite degree cover of M branched over L is a virtually fibred 3-manifold.*

The generic generalised Montesions link has type $\widetilde{SL_2}$. Our next theorem constructs an infinite family of such links which virtually fibre.

Theorem 1.10 *Let $p \geq 3$ be an odd integer. If n is a non-negative multiple of p, then the generalised Montesinos link $K(-g; e_0; \frac{q_1}{p}, \frac{q_2}{p}, ..., \frac{q_n}{p})$ is virtually fibred.*

The links described in this result are of type $\widetilde{SL_2}$ when $pe_0 \neq -(q_1 + q_2 + ... + q_n)$ and $(g, p, n) \notin \{(0, 3, 3), (0, p, 0), (1, p, 0), (2, p, 0)\}$ (c.f. Identities (2), (3) and Table 1).

Most of the links discussed in the theorems above are not fibred. For example, see [Ga2] for the classification of fibred pretzel links. Further, most classic Montesinos links are hyperbolic in the sense that their complements admit complete, finite volume hyperbolic structures. Indeed, Oertel has shown that a classic Montesinos link is either an atoroidal Seifert link (i.e. its exterior is atoroidal and admits a Seifert fibred structure), or a Montesinos link of type $\mathbb{E}^3$, or a hyperbolic link (see the proof of Corollary 5, [Oe]). Further, it can be shown that if $K(0; 0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$ is a Seifert link with $n \geq 3$, then it is a Montesinos link of spherical type (c.f. the proof of Corollary 1.4). In particular, all classic Montesinos links of type $Nil, \mathbb{H}^2 \times \mathbb{R}$, or $\widetilde{SL_2}$ are hyperbolic.

The proofs of the results discussed above use the geometric constraints on W_K imposed by the hypotheses in an essential way. That of Theorem 1.10 is the most delicate and

uses the strategy set out by Walsh. One shows that there is a cover of S^3, branched over the link in question, and a component of the branch set whose exterior admits a surface bundle structure with the property that all the other components of the branch set are transverse to its fibres. This implies the desired conclusion. In all cases, the construction of the fibrings is explicit enough for us to deduce a strong property of the fundamental groups of the associated manifolds.

A group is called *bi-orderable* if it admits a total order which is invariant under both left and right multiplication, and *virtually bi-orderable* if it has a finite index bi-orderable subgroup. Though the fundamental groups of 3-manifolds are rarely bi-orderable, it is possible that they are always virtually bi-orderable (c.f. [BRW]).

Theorem 1.11 *The fundamental groups of the exteriors of generalised Montesinos links which are not classic Montesinos links of type $\widetilde{SL_2}$ are virtually bi-orderable, as are the fundamental groups of the classic Montesinos links of type $\widetilde{SL_2}$ described in Theorem 1.10.*

Dale Rolfsen was independently aware of parts of this theorem.

We review several constructions and set our conventions in §2. The virtual fibring of links contained in essential surfaces is considered in §3 where Theorem 1.7 and Corollary 1.8 are proven. Section 4 considers the virtual fibring of generalised Montesinos links other than classic ones of type $\widetilde{SL_2}$ and in particular, Theorems 1.2 and 1.3 are dealt with there. The latter is applied in §5 to show certain branched covers of classic Montesinos links virtually fibre. The proofs of Corollaries 1.4, 1.5, and 1.9 are contained in this section. The final two sections of the paper deal with Theorems 1.10 and 1.11 respectively.

The authors wish to thank Michel Boileau for providing background material on Montesinos links.

2 Some constructions, conventions, and notation

We will use the notation introduced above as well as the following constructions and conventions throughout the paper.

Let $K = K(-g; e_0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$ be a generalised Montesinos link and $(W_K, \tilde{K}) \to (S^3, K)$ the 2-fold branched cover described in the introduction. We endow W_K with the Seifert fibred structure $f : W_K \to \mathcal{B}_K$ described by Montesinos ([Mo1], [Mo2]). In particular, $\tilde{K}$ has exactly g *vertical* components which are fibres of the given Seifert structure. Its remaining *horizontal* components intersect each Seifert fibre in either zero or two points. We use $\tilde{K}_v$

and $\tilde{K}_h$ to denote, respectively, the union of the vertical and horizontal components of $\tilde{K}$, and $K_h, K_v \subset K$ their images in S^3.

The orbifold $\mathcal{B}_K$ has topological form $(\#_g P^2)(p_1, p_2, \ldots, p_n)$ and the image of $\tilde{K}$ in $\mathcal{B}_K$ is the disjoint union of the g points $f(\tilde{K}_v)$ and a simple closed curve $K^* = f(\tilde{K}_h)$ which contains the cone points of $\mathcal{B}_K$. Number the cone points $c_1, \ldots, c_n$ respecting the order with which they occur on K^*. This can be done so that each c_i has order p_i.

The number of components $|K|$ of K is given by

$$|K| = \begin{cases} g+1 & \text{if each } p_i \text{ is odd and } (e_0 + q_1 + \cdots + q_n) \text{ is odd} \\ g+2 & \text{if each } p_i \text{ is odd and } (e_0 + q_1 + \cdots + q_n) \text{ is even} \\ g + \#\{i : p_i \text{ is even}\} & \text{if some } p_i \text{ is even} \end{cases} \quad (1)$$

while the Euler number of W_K and Euler characteristic of $\mathcal{B}_K$ are given by

$$e(W_K) = -(e_0 + \sum \frac{q_j}{p_j}) \quad (2)$$

$$\chi(\mathcal{B}_K) = 2 - g - n + \sum \frac{1}{p_j} \quad (3)$$

Fix a geometric structure on W_K such that the covering involution τ is an isometry and the branch set $\tilde{K} = \text{Fix}(\tau)$ is geodesic. There is an associated geometric structure on $\mathcal{B}_K$ such that the image of a geodesic in W_K is either a point or a geodesic. In particular, $K^* = f(\tilde{K}_h)$ is geodesic. If we define K_0^* to be the 1-orbifold obtained by cutting open K^* at the cone points of even order and thinking of the resulting endpoints as mirror points, then the map $f| : \tilde{K}_h \to K_0^*$ is an orbifold double cover.

All coverings of W_K and orbifold coverings of $\mathcal{B}_K$ will be endowed with the induced metrics. The Seifert fibres of W_K, or its covers, will be denoted by ϕ.

Lemma 2.1 *Let $g : V \to W_K$ be a smooth cover and $J_h = g^{-1}(\tilde{K}_h)$. Endow V with the geometric structure and Seifert fibring induced by g. If ϕ is a Seifert fibre of V, then wherever ϕ intersects J_h, the intersection is perpendicular.*

Proof. It suffices to verify that the lemma holds when $g = 1_{W_K}$.

Let $x \in \tilde{K}_h \cap \phi \subset W_K$ and $u, v \in T_x W_K$ be unit vectors tangent at x to $\tilde{K}_h$ and ϕ respectively. Clearly $d_x \tau(u) = u$ and since τ is an isometry of W_K which leaves ϕ invariant, $d_x \tau(v) = \pm v$. Since τ is an orientation-preserving isometry of W_K which is not the identity, we must have $d_x \tau(v) = -v$. Hence $\langle u, v \rangle_x = \langle d_x \tau(u), d_x \tau(v) \rangle_x = -\langle u, v \rangle_x$, which proves the lemma. $\diamond$

It follows from the proof of the lemma that τ reverses the orientation of any Seifert fibre of W_K which intersects some component of $\tilde{K}_h$.

If W_K has an infinite fundamental group, there is a finite regular cover $\Psi : Y \to W_K$ such that the induced Seifert structure on Y is a locally-trivial circle bundle over an orientable surface F and there is an induced orbifold cover $F \to \mathcal{B}_K$. Further, if $e(W_K) = 0$, then $Y = F \times S^1$ and Ψ is the projection $F \times S^1 \to F$. A convenient way to construct such covers is as follows.

It is well-known that 2-dimensional orbifolds with non-positive Euler characteristic admit finite regular orbifold covers which are surfaces (i.e. have no singular points). Fix such a cover $\psi : F \to \mathcal{B}_K$ and recall that there is an associated commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\hat{f}} & F \\
\Big\downarrow{\Psi} & & \Big\downarrow{\psi} \\
W_K & \xrightarrow{f} & \mathcal{B}_K
\end{array}
\tag{4}
$$

where
   $\cdot$ Ψ is a regular cover of the same degree as ψ (and the same group of deck transformations);
   $\cdot$ the Seifert structure on Y induced by Ψ has base orbifold F and $\hat{f}$ is the Seifert quotient map.
Note that $\hat{f} : Y \to F$ is a locally-trivial circle bundle and the generic fibre of Y is mapped homeomorphically to a fibre of W_K.

Set $L = \Psi^{-1}(\tilde{K}) \subset Y$ and $L^* = \psi^{-1}(K^*) \subset F$. By construction, L is a disjoint union of $L_h = \Psi^{-1}(\tilde{K}_h)$, a geodesic link in Y orthogonal to the fibres of the Seifert structure, and $L_v = \Psi^{-1}(\tilde{K}_v)$, a finite union of Seifert fibres of Y. Clearly, $L^* = \hat{f}(L_h)$. Hence L^* is a finite union of closed geodesics in F (a *geodesic 1-complex*).

$$\tag{5}$$

Lemma 2.2 *If $K = K(-g; e_0; \frac{q_1}{p_1}, ..., \frac{q_n}{p_n})$ is a generalised Montesinos link with $e(W_K) = 0$, then W_K admits the structure of a surface bundle or semi-bundle whose fibres are totally geodesic and everywhere orthogonal to the Seifert structure. Further, $\tilde{K}_h$ is contained in a finite union of surface fibres.*

Proof. First observe that W_K admits an $\mathbb{X}^2 \times \mathbb{R}$ structure where $\mathbb{X}^2 = \mathbb{S}^2, \mathbb{H}^2$, or $\mathbb{E}^2$ depending on whether $\chi^{orb}(S^2(p_1, \ldots, p_n))$ is positive, negative, or zero. This information is contained in Table 1 when $\chi^{orb}(S^2(p_1, \ldots, p_n)) \neq 0$ and in the discussion following Theorem 4.3 of [Sc] when $\chi^{orb}(S^2(p_1, \ldots, p_n)) = 0$. The surfaces $\mathbb{X}^2 \times \{*\} \subset \mathbb{X}^2 \times \mathbb{R}$ descend to give W the structure of a bundle $\mathcal{F}$ of closed totally geodesic surfaces over a 1-dimensional orbifold $\mathcal{O}^1$ whose leaves are perpendicular to the Seifert fibres (see §4 of [Sc]). Lemma 2.1 then shows that each component of $\tilde{K}_h$ is contained in some surface fibre of $\mathcal{F}$. $\diamond$

Remark 2.3 It follows from the proof that the generic surface fibre of $\mathcal{F}$ is a 2-sphere if $\chi(\mathcal{B}_K) > 0$, a torus if $\chi(\mathcal{B}_K) = 0$, and a hyperbolic surface $\chi(\mathcal{B}_K) < 0$.

In the remainder of the paper, homology and cohomology groups will be understood to have $\mathbb{Z}$ coefficients unless otherwise specified.

3 Virtual fibrings of links in essential surfaces

We prove Theorem 1.7 and Corollary 1.8 in this section.

Lemma 3.1 *Let M be an oriented 3-manifold, $L \subset M$ an oriented link, $\alpha \in H^1(M)$ a cohomology class which is non-zero on each component of L. Then there exists a closed 1-form ω such that $[\omega] = \alpha$, and such that $\omega_{|TL}$ is non-zero. That is, for each $v \neq 0$ tangent to a component of L, $\omega(v) \neq 0$.*

Proof. Let $\mathcal{N}(L)$ be a regular neighborhood of L diffeomorphic to $L \times D^2$ and fix a smooth map $p : M \to S^1$ such that α is represented by $p^*\eta$ where $\eta \in \Lambda^1(S^1)$ is a 1-form representing the fundamental class of S^1. Since α is non-zero on each component of L, we can homotope $p_{|L}$ to a smooth covering map $f : L \to S^1$. Let $f' : \mathcal{N}(L) \to S^1$ be the composition $\mathcal{N}(L) = L \times D^2 \xrightarrow{\text{proj}} L \xrightarrow{f} S^1$. Since the inclusion $\mathcal{N}(L) \to M$ is a cofibration, p is homotopic to a map $F : M \to S^1$ such that $F_{|\mathcal{N}(L)} = f'$. Finally, if we perturb F rel $(L \times \frac{1}{2}D^2)$ to a smooth map $F' : M \to S^1$, then $\omega = (F')^*\eta$ is a closed 1-form on M which represents α and which is nowhere vanishing on TL. $\diamond$

Proof of Theorem 1.7. Since every component of L is homologically non-trivial in $H_1(M; \mathbb{Q})$, we may find a cohomology class $\alpha \in H^1(M)$ which is non-vanishing on each such component. By Lemma 3.1, there is a closed 1-form $\omega \in \Lambda^1(M)$ representing α which is nowhere vanishing on the tangent vectors to L.

Let $\eta \in \Lambda^1(S^1)$ be a nowhere zero closed 1-form on S^1 representing the fundamental class of $H^1(S^1)$. Then $p^*\eta$ vanishes on the tangent space of each fibre of p, and so in particular, it vanishes on all tangent vectors to L. Note, moreover, that as M is compact, there is an integer $N \gg 0$ such that $\beta = \omega + Np^*\eta$ is a nowhere vanishing closed 1-form which represents an integral cohomology class and whose restriction to TL is also nowhere vanishing. Thus, kernel(β) represents a plane field tangent to a fibring of $M \to S^1$ ([Ti]) such that the fibres are everywhere transverse to L. The remaining conclusions of the theorem follow immediately from this. $\diamondsuit$

We complete this section with the proof of Corollary 1.8. We begin with a lemma.

Lemma 3.2 *Let F be a compact, connected, orientable surface and $L \subset F \times S^1$ a link contained in the interior of a finite union of the surface fibres $F \times \{x\}$. Let $L_1, L_2, \ldots, L_m$ be the components L and $L_1^*, L_2^*, \ldots, L_m^*$ their projection in F. If each L_i^* is homotopically non-trivial in F, there is a finite connected cover $\psi : \tilde{F} \to F$ such that each component of the inverse image of each L_i^* in $\tilde{F}$ is non-trivial in $H_1(\tilde{F})$.*

Proof. Take $p : \tilde{F} \to F$ to be the cover corresponding to the homomorphism $\pi_1(F) \to H_1(F; \mathbb{Z}/2)$. We claim that each component of the inverse image by p of an essential simple closed curve $C \subset F$ in homologically non-trivial in $\tilde{F}$. To see this, first note that given C it suffices to find a 2-fold cover of F to which C lifts to homologically non-trivial curves, since p factors through any such cover. If C is homologically non-trivial in F, any 2-fold cover will do. If C is homologically trivial, we have $F = F_0 \cup F_1$ where $\partial F_0 = C \subset \partial F_1$. The essential nature of C implies that $H_1(F_0; \mathbb{Z}/2), H_1(F_1; \mathbb{Z}/2) \neq 0$. If $\partial F_1 = C$, we can find a homomorphism $\pi_1(F) \to \mathbb{Z}/2$ which induces epimorphisms $\pi_1(F_j) \to \mathbb{Z}/2$ for $j = 0, 1$ and it is easy to see that the associated cover is of the desired form. On the other hand, if $\partial F_1 \neq C$, there is a homomorphism $\pi_1(F) \to \mathbb{Z}/2$ which induces an epimorphism $\pi_1(F_0) \to \mathbb{Z}/2$ and the zero homomorphism $\pi_1(F_1) \to \mathbb{Z}/2$. Then each a lift of C to the associated cover separates distinct boundary components of the underlying surface, so we are done. $\diamondsuit$

Our next result follows immediately from Lemma 3.2 and Theorem 1.7.

Corollary 3.3 *Let F be a compact, connected, orientable surface and $L \subset F \times S^1$ a link contained in the interior of a finite union of the surface fibres $F \times \{x\}$. If the projection*

of each component of L is homotopically non-trivial in F, there is a finite degree cover $\Psi : Y = \tilde{F} \times S^1 \to F \times S^1$ and a surface bundle structure on Y which is everywhere transverse to $\tilde{L} = \Psi^{-1}(L)$. In particular if F is closed, the boundary of any fibre in the induced bundle structure on the exterior of $\tilde{L}$ consists of meridians of $\tilde{L}$. ◇

Proof of Corollary 1.8. Let W be a connected, irreducible, orientable Seifert manifold, F a not necessarily connected essential surface in W, and $\emptyset \neq L \subset \text{int}(F)$ a link each component of which is homotopically essential in F. Then no component of F is a sphere or disc. Since W is irreducible, we can suppose, after an isotopy, that the components of F are either all horizontal or all vertical. In the former case, there is a finite cyclic cover $Y = F_0 \times S^1 \to W$ where F_0 is a component of F. The inverse image of F in Y is a union of fibres $F_0 \times \{*\}$ which contains the inverse image $\hat{L}$ of L. Since each component of $\hat{L}$ is homotopically non-trivial in the surface fibre which contains it, Lemma 3.2 implies that there is a finite connected cover $\tilde{F}_0 \to F_0$ such that each component of the inverse image $\tilde{L}$ of $\hat{L}$ in $\tilde{Y} = \tilde{F}_0 \times S^1$ (under the natural covering map $\tilde{Y} \to Y$) is homologically non-trivial in the surface fibre $\tilde{F}_0 \times \{*\}$ which contains it. Theorem 1.7 then shows that there is a product structure on $\tilde{Y}$ such that $\tilde{L}$ is everywhere transverse to the surface fibres, which implies the desired result.

Now suppose that F is a finite union of vertical surfaces. Then $L \subset F_1 \cup \ldots \cup F_n$ where each F_i is a vertical torus or annulus. We can suppose that for each i, $L \cap F_i \neq \emptyset$ and so is a finite collection of parallel curves in F_i. Let $\mathcal{N}(F) = \mathcal{N}(F_1) \cup \mathcal{N}(F_2) \cup \ldots \cup \mathcal{N}(F_n)$ be a fibred tubular neighbourhood of F. That is, $\mathcal{N}(F)$ is a union of Seifert fibres of W. The Seifert structure on each $\mathcal{N}(F_i)$ can be altered so that $L \cap F_i$ is a union of fibres. Since $\overline{W \setminus \mathcal{N}(F)}$ inherits a Seifert structure from W, we see that $W \setminus L = (\overline{W \setminus \mathcal{N}(F)}) \cup (\mathcal{N}(F_1) \setminus (L \cap F_1)) \cup \ldots \cup (\mathcal{N}(F_n) \setminus (L \cap F_n))$ is a graph manifold with non-empty boundary. The main result of [WY] states that such a manifold virtually fibres, which completes the proof of the corollary. ◇

4   Virtual fibrings of Montesinos links other than classic ones of type $\widetilde{SL_2}$

In this section we prove Theorems 1.2 and 1.3. To that end, let $K = K(-g; e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, ..., \frac{q_n}{p_n})$ be a generalised Montesinos link which is not of type $\widetilde{SL_2}$. Given Walsh's result (Theorem 1.1) we need only consider the cases

- K has type $\mathbb{S}^2 \times \mathbb{R}$
- K has type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$

- K has type Nil

- K is a non-classic generalised Montesinos link of type $\widetilde{SL_2}$

These will be dealt with individually in the following four subsections.

Remark 4.1 If $n = 0$ but $(g, e_0) \neq (0, 0)$, the exterior of K is Seifert fibred for $g = 1$ and a graph manifold in general (c.f. Figure 1). Thus it is virtually fibred by [WY].

4.1 K has type $\mathbb{S}^2 \times \mathbb{R}$

In this case we must prove Theorem 1.2.

We can assume that $e_0 = 0$ when $n > 0$. Then since $\chi(\mathcal{B}_K) > 0$ and $e(W_K) = 0$, Identities (2) and (3) imply that either (a) $g = e_0 = n = 0$ or (b) $g = 0, n = 2$, and $\frac{q_1}{p_1} = -\frac{q_2}{p_2}$, or (c) $g = 1$ and $n = 0$. The latter case can be dealt with using Remark 4.1, but it is simpler to note that K is the connected sum of two Hopf links and so its exterior fibres over the circle with fibre a pair of pants. In case (a), K is a trivial link of two components. Since the exterior of this link is reducible, it cannot be virtually fibred.

Finally consider case (b). Identity (1) shows that K has two components. By applying Lemma 2.2 and the remark which follows its proof, we see that W_K admits the structure of a bundle or semi-bundle $\mathcal{F}$ whose generic fibre is a totally geodesic 2-sphere everywhere orthogonal to the Seifert structure. Further, $\tilde{K} = \tilde{K}_h$ is contained in a finite union of surface fibres.

Since $\mathcal{B}_K = S^2(p_1, p_1)$ is orientable, the Seifert fibres of W_K can be coherently oriented and therefore so can the surface fibres of W_K. Hence W_K fibres over S^1. We noted after the proof of Lemma 2.1 that the covering involution τ reverses the orientation of the Seifert fibres which intersect $\tilde{K}_h = \tilde{K}$, and so it induces an orientation reversing involution $\bar{\tau} : S^1 \to S^1$. Hence there are precisely two τ-invariant leaves F_1, F_2 of $\mathcal{F}$. Clearly $\tilde{K} = \text{Fix}(\tau) \subset F_1 \cup F_2$. Since $\tau|F_j$ reverses orientation, $\text{Fix}(\tau) \cap F_j$ is a circle and $F_0/\tau, F_1/\tau \subset S^3$ are disjoint 2-discs whose union has boundary K. Thus K is a trivial link of two components, and so as above, it cannot virtually fibre.

4.2 K has type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$

We must prove Theorem 1.2 in this case and Theorem 1.3 under the additional assumption that K is a classic Montesinos link.

If K is a generalised Montesinos of type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$, then W_K admits the structure of a surface bundle or semi-bundle whose fibres are totally geodesic surfaces everywhere orthogonal to the Seifert structure (Lemma 2.2). Further, $\tilde{K}_h$ is contained in a finite union of surface fibres (or semi-fibres). Since $e(W_K) = 0$, there is a finite cover $\Psi : Y = F \times S^1 \to W_K$ such that the circles $\{x\} \times S^1$ are Seifert fibres of the induced Seifert structure on Y and $L_h = \Psi^{-1}(\tilde{K}_h)$ is contained in a finite union of surface fibres of the form $F \times \{x\}$. Since each component of L_h is geodesic in Y, it is homotopically essential in the surface fibre which contains it. By construction, $L_v = \Psi^{-1}(\tilde{K}_v)$ is a finite union of Seifert fibres of Y, and hence its exterior admits an induced product structure $F_0 \times S^1$ in which each component of L_h is a homotopically essential loop in a surface fibre $F_0 \times \{x\}$. Corollary 1.8 can now be applied to complete the proof of Theorem 1.2 in the case we are considering. Further, when K is a classic Montesinos link, $L_v = \emptyset$ and so $F_0 = F$ is closed. Hence Corollary 3.3 implies that Theorem 1.3 holds for classic Montesinos links of type $\mathbb{E}^3$ or $\mathbb{H}^2 \times \mathbb{R}$.

4.3 K has type Nil

We prove Theorem 1.2 in this case and Theorem 1.3 under the additional assumption that K is a classic Montesinos link. This will complete the proof of the latter theorem.

When K has type Nil, $\mathcal{B}_K$ is Euclidean, so if $F \to \mathcal{B}_K$ is the cover discussed in §2 (see Diagram 4), then F is a torus. In particular, its closed geodesics are simple. Choose a geodesic α on F whose slope is distinct from the slopes of the closed geodesics whose union is L^* and note that F fibres over the circle with every fibre a geodesic parallel to α. Then L^* is transverse to these fibres and since the restriction of $\hat{f} : L \to L^*$ is a cover away from the non-manifold points of L^*, L_h is transverse to the fibres of the induced torus bundle structure on Y. Hence the exterior of L_h in Y has an induced surface bundle structure $\mathcal{F}$ whose fibres have genus 1 and boundary components which are meridians of the components of L_h. Note that this completes the proof of Theorem 1.3 when K is a classic Montesinos link of type Nil, for then $L = L_h$. It also reproves this theorem when K is a classic Montesinos link of type $\mathbb{E}^3$.

To complete the proof of Theorem 1.2, suppose that K is a generalised Montesinos link of type Nil with $g > 0$. We can suppose that $n > 0$ by Remark 4.1 and therefore we can arrange for $e_0 = 0$. Identity (3) then implies that $\mathcal{B}_K = P^2(2, 2)$, so $g = 1$. Thus $K = K(-1; 0; \frac{a}{2}, \frac{b}{2})$ for some odd integers a, b. On the other hand, since $g = 1$, we can twist along a meridian disk in the exterior of the unknot K_v to construct a homeomorphism from the exterior of K to that of either $K(-1; 0; \frac{1}{2}, \frac{1}{2})$ or $K(-1; 0; \frac{1}{2}, \frac{-1}{2})$. In the first case the exterior of K homeomorphic to a 2-fold cover of the Whitehead link exterior (see Figure 3)

and so fibres over S^1, while in the second, its exterior is homeomorphic to the exterior of a generalized Montesinos link of type $\mathbb{E}^3$ (c.f Identities (2), (3) and Table 1), and therefore virtually fibres.
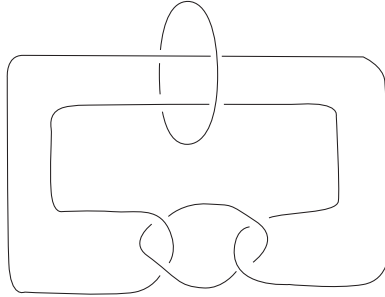

Figure 3: The link $K(-1; 0; \frac{1}{2}, \frac{1}{2})$

4.4 K is a non-classic generalised Montesinos link of type $\widetilde{SL_2}$

We prove Theorem 1.2 in this case. Let $K = K(-g; e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, ..., \frac{q_n}{p_n})$ where $g > 0$. The g components of K_v are Seifert fibres in the (orbifold) Seifert structure on S^3 with singular set the classic Montesinos link K_h and cone angle π ([Mo1], [Mo2]). Thus the 2-fold cover $W^0_{K_h}$ of the exterior of K_v branched over K_h is Seifert fibred with non-empty boundary and base orbifold $\mathcal{B}^0_{K_h}$ whose underlying space is a $2g$-punctured 2-sphere and n cone points of order $p_1, p_2, \ldots, p_n$. Hence $\chi(\mathcal{B}^0_{K_h}) = \chi(\mathcal{B}_K) - g < 0$ and so as $\partial W^0_{K_h} \neq \emptyset$, $W^0_{K_h}$ admits an $\mathbb{H}^2 \times \mathbb{R}$ structure. An argument like that used in the proof of Lemma 2.2 shows that $W^0_{K_h}$ admits a fibring over a 1-dimensional orbifold whose fibres are perpendicular to the Seifert structure. Further, $\tilde{K}_h$ is contained in a finite union of surface fibres. Thus there is a finite cover $F \times S^1 \to W^0_{K_h}$ such that the inverse image L of $\tilde{K}_h$ in $F \times S^1$ is contained in a finite union of sets of the form $F \times \{x\}$. Further, each component of L is homotopically essential in the surface $F \times \{x\}$ which contains it. Corollary 3.3 implies that K virtually fibres. This completes the proof of Theorem 1.2. $\diamond$

Remark 4.2 It follows from the arguments above and Remark 2.3 that for the constructed virtual fibring of a generalised Montesinos link exterior, the genus of a surface fibre is 0 when K has type $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$, and 1 when K has type $\mathbb{E}^3$ or Nil.

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5 Virtually fibred branched covers

In this section we prove Corollary 1.4, Corollary 1.5, and Corollary 1.9.

Proof of Corollary 1.4. Let $K = K(0; e_0; \frac{q_1}{p_1}, \frac{q_2}{p_2}, ..., \frac{q_n}{p_n})$ be a classic Montesinos link of type $\mathbb{E}^3, Nil$, or $\mathbb{H}^2 \times \mathbb{R}$ and $Z \to W_K$ a cover branched over $\tilde{K}$. According to Theorem 1.3, there is a cover $Y \to W_K$ where Y is a surface bundle over the circle whose fibres are transverse to the inverse image L of $\tilde{K}$. Hence, the inverse image of this surface fibration in any cover of Y, branched over L, is also a surface bundle. Since the cover $\tilde{Z}$ of Z induced by $Y \to W_K$ admits such a branched cover to Y, Z virtually fibres over the circle. $\diamond$

Proof of Corollary 1.5. It is well known that for $m \geq 3$ the $2m$-fold cyclic covers $\Sigma_{2m}(K) \to S^3$ branched over a hyperbolic knot K are hyperbolic [Du] and represent infinitely many different commensurability classes. To see the latter, let M_K denote the exterior of K and μ its meridinal slope. The orbifold $M_K(\mu^{2m})$ obtained by μ^{2m} Dehn surgery on K lies in the same commensurability class as $\Sigma_{2m}(K)$, and so has the same invariant trace field. Since $\pi_1(M_K(\mu^{2m}))$ has elements of order $2m$ and its invariant trace field is generated the traces of the squares of its elements, this number field contains $\cos(\frac{2\pi}{m})$. Since the degree of the latter over $\mathbb{Q}$ tends to infinity with m, there are infinitely many different invariant trace fields associated to the $\Sigma_{2m}(K)$, which implies the desired conclusion.

Given Corollary 1.4, the proof of Corollary 1.5 will be completed if we show that classic Montesinos links of type Nil and $\mathbb{H}^2 \times \mathbb{R}$ are hyperbolic, for in this case the associated branched covers are hyperbolic manifolds by [Du].

Suppose that K is a classic Montesinos link of type Nil or $\mathbb{H}^2 \times \mathbb{R}$. Then $\mathcal{B}_K = S^2(p_1, \ldots, p_n)$ where $n \geq 3$ and $(p_1, \ldots, p_n)$ is not a Platonic triple. In this case, the fundamental group of W_K is infinite, non-Abelian, and its centre is infinite cyclic and generated by the class ζ represented by a regular fibre ϕ.

We remarked in the introduction that a classic Montesinos link is either a hyperbolic link, an $\mathbb{E}^3$ classic Montesinos link, or an atoroidal Seifert link. In the latter case, the reader will verify that K is the union of at most three fibres of some Seifert structure on S^3 and further, the sum of the number of components of K and the number of exceptional fibres in the induced Seifert structure on the exterior of K is at most three. The latter Seifert structure induces a τ-invariant one on W_K having base orbifold a 2-sphere with at most three cone points. Further, τ preserves the orientation of the associated Seifert fibres. Since $\pi_1(W_K)$ is non-Abelian, there are precisely three cone points in this structure, and so, as above, the centre of $\pi_1(W_K)$ is generated by the class ζ' represented by a regular fibre ϕ' of this structure. We can assume that the orientation of ϕ' is chosen so that $\zeta' = \zeta$. On the

other hand, τ reverses the orientation of the Montesinos Seifert fibres ϕ, and therefore

$$\zeta^{-1} = \tau_\#(\zeta) = \tau_\#(\zeta') = \zeta' = \zeta$$

Thus $\zeta^2 = 1$ and so as W_K is irreducible, $\pi_1(W_K)$ is finite. In other words, K has spherical type, contrary to our assumption. Thus Nil and $\mathbb{H}^2 \times \mathbb{R}$ classic Montesinos links are hyperbolic. This completes the proof of Corollary 1.5. $\diamond$

Proof of Corollary 1.9. This corollary can be deduced using an argument similar to that used in the proof of Corollary 1.4. $\diamond$

6   An infinite family of virtually fibred generalised Montesinos links of type $\widetilde{SL_2}$

The goal of this section is to prove Theorem 1.10. To that end let $K = K(-g; e_0; \frac{q_1}{p}, \frac{q_2}{p}, ..., \frac{q_n}{p})$ be a generalised Montesinos link where $p \geq 3$ is odd and n is a non-negative multiple of p. By Remark 4.1 we can suppose that $n > 0$ and therefore that $e_0 = 0$. We also assume that we are not in the situation $p = n = 3$ or $e(W_K) = 0$ or $g > 0$ as those cases have been handled in Theorem 1.2. Thus K is a classic Montesinos link of type $\widetilde{SL_2}$. It follows from Identity (1) that K has at most two components. We shall prove Theorem 1.10 in detail in the case where it is a knot (§6.1) and indicate afterwards what changes are necessary to deal with the case that it has two components (§6.2).

6.1 K is a knot

In this case K is a classic Montesinos knot so $K = K_h$. The Seifert structure is $f : W_K \to S^2(p, \ldots, p)$ and the branched cover $(W_K, \tilde{K}) \to (S^3, K)$ where $K^* = f(\tilde{K})$ is a geodesic equator of $S^2(p, p, \ldots, p)$ which passes through each of the cone points (c.f. Figure 4). Since p is odd, $f| : \tilde{K} \to K^*$ is a double cover.

The group $\Gamma = \pi_1(S^2(p, \ldots, p))$ has a presentation

$$\Gamma = \langle x_1, \ldots, x_n \mid x_1^p = \ldots = x_n^p = x_1 x_2 \ldots x_n = 1 \rangle$$

where x_i is represented by a small circular loop in $S^2(p, \ldots, p)$ centered at c_i. Let $\psi : F \to S^2(p, \ldots, p)$ be the p-fold cyclic orbifold cover of $S^2(p, ..., p)$ corresponding to the homomorphism

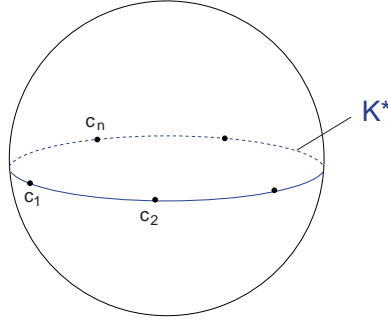$$h : \Gamma \longrightarrow \mathbb{Z}/p, x_k \mapsto \bar{1} \text{ for each } k.$$

Figure 4: The orbifold $S^2(p, ..., p)$ with n cone points $c_1, ..., c_n$, each with order p.

This homomorphism takes each x_k to an element of order p and therefore F is a closed orientable surface without cone points. Further, $\psi^{-1}(c_i)$ is a point $\hat{c}_i \in F$ for $1 \leq i \leq n$.

Let $\Psi : Y \to W_K$ be the associated p-fold cyclic cover (c.f. §2) and set $L = \Psi^{-1}(\tilde{K})$, a geodesic link in Y. The composition $\pi_1(W_K) \to \pi_1(S^2(p, \ldots, p)) \xrightarrow{h} \mathbb{Z}/p$ factors through $H_1(W_K)$ and since $\tilde{K}$ is homologically trivial in W_K (it bounds the lift of a Seifert surface for K), the image of a fundamental class of $\tilde{K}$ is sent to 0 under the composition. Thus L has p components:

$$L = L_1 \cup L_2 \cup \ldots \cup L_p.$$

Recall $L^* = \hat{f}(L)(= \psi^{-1}(K^*))$ so that

$$L^* = L_1^* \cup \ldots \cup L_p^*$$

where $L_i^* = \hat{f}(L_i)$ is a geodesic in F. Since p is odd, each L_j^* contains $\{\hat{c}_1, \ldots, \hat{c}_n\}$. Further, L_i^* is simple. To see this, first note that as $f| : K \to K^*$ is a 2-fold cover, $2[K^*] = 0 \in H_1(\Gamma) \cong (\mathbb{Z}/p)^{n-1}$. Since p is odd it follows that $[K^*] = 0 \in H_1(\Gamma)$. Thus $\psi| : L_j^* \to K^*$ is a homeomorphism and thus $f| : L_j \to L_j^*$ is a 2-fold cover. Now L_i^* passes successively through the $\hat{c}_i$. If the first return to $\hat{c}_1$ along L_j^* is at an angle $2\pi j/p$, then following twice around you come in at an angle of $4\pi j/p$. But going twice around brings you back to the beginning, so $4\pi j/p = 2k\pi$ for some integer k. Thus $kp = 2j$ so that $2\pi j/p$ is either π or 2π. The first case is impossible, so $j = p$, which yields the desired conclusion.

Denote by γ the deck transformation of Ψ corresponding to $\bar{1} \in \mathbb{Z}/p$ and let γ^* be the corresponding action on F. By construction, $\text{Fix}(\gamma^*) = \{\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n\}$ and F admits an orientation such that γ^* is rotation by $2\pi/p$ near each $\hat{c}_i$. We may suppose that $\gamma(L_j) = L_{j+1}$ and $\gamma^*(L_j^*) = L_{j+1}^*$ for each $j \pmod p$. If we fix an orientation for L_1^* and give $L_j^* = (\gamma^*)^{j-1}(L_1^*)$ the induced orientation, then L_j^* intersects L_1^* in an angle of $\frac{2\pi j}{p}$ at each of the points $\hat{c}_i$. Thus each L_j^* is a non-separating curve in F and L_j is transverse to
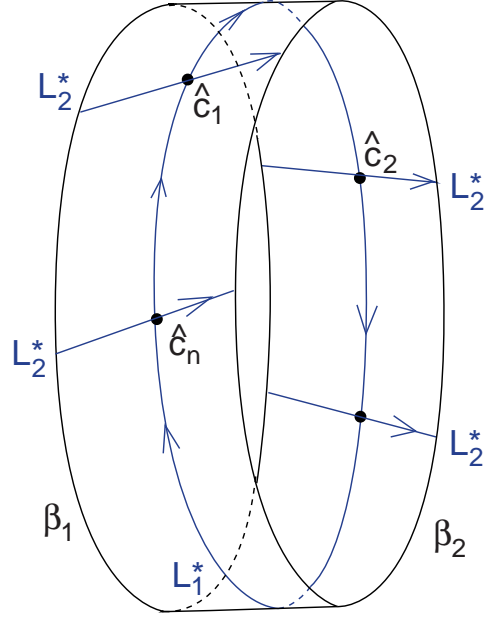
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Figure 5: A neighborhood of L_1^* in the surface F.

$T = \hat{f}^{-1}(L_1^*) \subset Y$ for $2 \leq j \leq p$. The situation near L_1^* in F is depicted in Figure 5.

By construction, T is a totally geodesic torus which inherits a Euclidean structure from the $\widetilde{SL_2}$ structure on Y. In particular, given any pair of closed, simple geodesics $\gamma_1, \gamma_2 \subset T$ which carry a basis for $H_1(T)$, there is an identification of T with $S^1 \times S^1$ where each $S^1 \times \{*\}$ is a geodesic isotopic to γ_1 and each $\{*\} \times S^1$ is a geodesic isotopic to γ_2. With respect to the product metric on $S^1 \times S^1$, the identification scales the jth factor by the length of γ_j.

Proposition 6.1 *The exterior of L_1 in Y is a surface semi-bundle.*

Proof. To prove this, we apply [WY].

Let R_2 be the closed ϵ-neighborhood of L_1^* in F (ϵ small) and R_1 the closure of the complement of R_2 in F. Denote by β_1, β_2 the two boundary components of R_2 (c.f. Figure 5) and define T_1 and T_2 to be the vertical tori in Y lying over β_1 and β_2 respectively. Let Y_i be the submanifold of Y lying over R_i ($i = 1, 2$) and set

$$Y_0 = Y_1 \cup_{T_1} Y_2$$

$$R_0 = R_1 \cup_{\beta_1} R_2.$$

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The restriction of the Seifert fibring of Y to each Y_i is a trivial circle bundle over R_i. We denote the fibres of any of these bundles by ϕ.

Choose a horizontal section B_0 of $Y_0 \to R_0$ so that $B_0 \cap T$ is geodesic and set $B_i = B_0 \cap Y_i$ ($i = 1, 2$). Since $B_2 \cap T$ and ϕ, a fibre of Y contained in T, carry a basis for $H_1(T)$, there is an identification $T = S^1 \times S^1$ (c.f. the paragraph preceding this proposition) such that $S^1 \times \{1\} = B_2 \cap T$ and for each $* \in S^1$, $S^1 \times \{*\}$ is a geodesic isotopic to $B_2 \cap T$ and $\{*\} \times S^1$ is geodesic fibre of $Y \to F$.

By construction, Y_2 is an ϵ neighborhood of T in Y. Identify it with $T \times [-\epsilon, \epsilon]$ where $T = T \times \{0\}$ and for $x \in T$, $\{x\} \times [-\epsilon, \epsilon]$ gives the unit speed parameterisation of the normal geodesic to T at x. Without loss of generality we suppose that $B_2 = (B_2 \cap T) \times [-\epsilon, \epsilon]$.

We shall call an arc $\sigma \times \{*\} \subset T \times \{*\}$ *straight* if σ is geodesic in T. Each fibre of $Y_2 \to R_2$ is straight.

Orient R_0 and let R_1 and R_2 and their boundaries have the induced orientations. Equip B_0, B_1, B_2 and their boundaries with the induced orientations. Next orient the circle fibres of Y_0 consistently and fix two such fibres ϕ_1, ϕ_2 on T_1, T_2 respectively. We shall write $T_{i,j}$ for T_j when we think of it as a boundary component of Y_i. Similarly we write $\phi_{i.j}$ for ϕ_j when we think of it as lying in $T_{i,j}$. Set $\alpha_{i,j} = B_i \cap T_{i,j}$, the boundary component of B_i on $T_{i,j}$. Note that (abusing notation) $\{\alpha_{i,j}, \phi_{i,j}\}$ forms a basis of $H_1(T_{i,j})$. As *homology classes in Y_i*, we have $\alpha_{i,1} = -\alpha_{i,2}$ and $\phi_{i,1} = \phi_{i,2}$.

The Seifert manifold Y is obtained from Y_1 and Y_2 by gluing $T_{1,1}$ to $T_{2,1}$ and $T_{1,2}$ to $T_{2,2}$ using the maps $g_j : T_{2,j} \to T_{1,j}$ determined by the conditions

$$(g_1)_*(\alpha_{2,1}) = -\alpha_{1,1} \qquad (g_1)_*(\phi_{2,1}) = \phi_{1,1}$$
$$(g_2)_*(\alpha_{2,2}) = -\alpha_{1,2} + e\phi_{1,2} \qquad (g_2)_*(\phi_{2,2}) = \phi_{1,2}$$

where $e \in \mathbb{Z}$ is the Euler number of the oriented circle bundle $Y \to F$. The naturality properties of Euler numbers imply that

$$|e| = p|e(W_K)| = p|\sum_j \frac{q_j}{p_j}| = |q_1 + q_2 + \ldots + q_n|$$

and therefore as p is odd and K has only one component, e is an odd integer (c.f. Identity (1)).

Note that T is fibred by geodesics isotopic to L_1, and this determines a fibring of $Y_2 = T \times [-\epsilon, \epsilon]$ by circles. Since L_1 has slope in T different from that of the fibres of Y, this leads to a new circle fibration $Y_2 \to \bar{R}_2$ with $\bar{R}_2$ an annulus. We call the fibring of Y coming from its Seifert structure the *original fibration* and its fibres, denoted ϕ, *original fibres.* In the same vein, $Y_2 \to \bar{R}_2$ will be called the *new fibration* and its fibres, denoted $\bar{\phi}$,

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new fibres. Subsets of Y which are vertical in the original fibration will be called ϕ-vertical. Subsets of Y_2 which are vertical in the new fibration will be called $\bar{\phi}$-vertical.

Choose a section of $Y_2 \to \bar{R}_2$ whose image $\bar{B}_2$ intersects T in a geodesic. This gives a new product decomposition $T = S^1 \times S^1$ where $S^1 \times \{1\} = \bar{B}_2 \cap T$ and $\{*\} \times S^1$ is a new fibre for each $* \in S^1$ (c.f. the remarks prior to the statement of this proposition). It also gives a new product decomposition $Y_2 = \bar{R}_2 \times S^1$.

Let N be a regular neighborhood of L_1 in Y_2, disjoint from $L_2 \cup \ldots \cup L_p$, which consists of new fibres. We denote the exterior of L_1 in Y_2 by M_2 and realize it concretely as $M_2 = Y_2 \setminus \text{int}(N)$. The boundary of M_2 consists of the tori $T_{2,1}, T_{2,2}$ and a third one $T_{2,3} = \partial N$. Now take M to be the exterior of L_1 in Y realized as $M = Y_1 \cup M_2$. The new fibration of Y_2 yields a decomposition $M_2 = \bar{R}_2^0 \times S^1$ where $\bar{R}_2^0 \subset \bar{R}_2$ is a once punctured annulus. Hence M is a graph manifold whose characteristic tori are T_1 and T_2.

Let $\bar{B}_2^0$ be the image of a section of $M_2 \to \bar{R}_2^0$, oriented arbitrarily, and endow $\partial \bar{B}_2^0$ with the induced orientation. Fix an orientation for the circle factor $\bar{\phi}$ of M_2.

Denote by $\bar{\alpha}_{2,j}$ the component of $\partial \bar{B}_2^0$ on $T_{2,j}$ $(j = 1, 2)$ and let $\bar{\alpha}_{2,3}$ be the remaining component. Finally let $\bar{\phi}_{2,j}$ be a fixed new fibre with the given orientation in $T_{2,j}$ $(j = 1, 2, 3)$. Then $\{\bar{\alpha}_{2,j}, \bar{\phi}_{2,j}\}$ forms a basis for $H_1(T_{2,j})$ $(j = 1, 2, 3)$. The relation between the old basis $\{\alpha_{2,1}, \phi_{2,1}\}$ of $H_1(T_{2,1})$ and the new one $\{\bar{\alpha}_{2,1}, \bar{\phi}_{2,1}\}$ is given by

$$\bar{\alpha}_{2,1} = a\alpha_{2,1} + b\phi_{2,1} \qquad \bar{\phi}_{2,1} = c\alpha_{2,1} + d\phi_{2,1}$$

where a, b, c, d are integers satisfying $ad - bc = \pm 1$. By reversing the orientation of the fibres ϕ if necessary, we may suppose that $ad - bc = 1$. Note that $c \neq 0$ since the slope of L_1 on T is different from that of the original fibre. We can assume that $c > 0$ while maintaining the conditions above by similtaneously changing, if necessary, the orientation of $\bar{B}_2^0$ and the new fibres. In fact, $c = 2$ since L_1 double covers L_1^*, though we won't use this fact till later in the proof.

As homology classes in Y_2, we have $\bar{\alpha}_{2,1} = -\bar{\alpha}_{2,2}$ and $\bar{\phi}_{2,1} = \bar{\phi}_{2,2}$. Thus

$$\bar{\alpha}_{2,2} = a\alpha_{2,2} - b\phi_{2,2} \qquad \bar{\phi}_{2,2} = -c\alpha_{2,2} + d\phi_{2,2}$$

Hence, with respect to the bases $\{\alpha_{1,j}, \phi_{1,j}\}$ of $H_1(T_{1,j})$ and $\{\bar{\alpha}_{2,j}, \bar{\phi}_{2,j}\}$ of $H_1(T_{2,j})$ $(j = 1, 2)$, the gluing maps $g_1 : T_{2,1} \to T_{1,1}$ and $g_2 : T_{2,2} \to T_{1,2}$ can be expressed as

$$(g_1)_*(\bar{\alpha}_{2,1}) = -a\alpha_{1,1} + b\phi_{1,1} \qquad\qquad (g_1)_*(\bar{\phi}_{2,1}) = -c\alpha_{1,1} + d\phi_{1,1}$$
$$(g_2)_*(\bar{\alpha}_{2,2}) = -a\alpha_{1,2} + (ae - b)\phi_{1,2} \qquad (g_2)_*(\bar{\phi}_{2,2}) = c\alpha_{1,2} + (d - ce)\phi_{1,2}.$$

The associated matrices are

$$G_1 = (g_1)_* = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix} \qquad G_2 = (g_2)_* = \begin{pmatrix} -a & ae - b \\ c & d - ce \end{pmatrix}.$$

with inverses

$$G_1^{-1} = \begin{pmatrix} -d & b \\ -c & a \end{pmatrix} \qquad G_2^{-1} = \begin{pmatrix} ce - d & ae - b \\ c & a \end{pmatrix}$$

We claim that the method of [WY] can be used to show that there are essential horizontal surfaces H_1 and H_2 contained in Y_1 and M_2, which piece together to form an essential surface H in the graph manifold M. Further, H is non-orientable so that M is the total space of a semi-bundle. We will provide some of the details, though refer the reader to [WY] for more complete explanations.

First we calculate the 2×2 matrix $Y = (y_{ij})$ and the 2×2 diagonal matrix $Z = \mathrm{diag}(z_1, z_2)$ defined on page 450 of [WY]. The graph of the JSJ-decomposition of M consists of two vertices v_1 and v_2, corresponding to the pieces Y_1 and M_2, and two edges correspond-ing to T_1 and T_2. Thus, referring to [WY] shows that $y_{11} = y_{22} = 0$, $y_{12} = y_{21} = \dfrac{2}{c}$, while $z_1 = \dfrac{d}{c} + \dfrac{ce - d}{c} = e$, $z_2 = \dfrac{a}{c} + \dfrac{-a}{c} = 0$. The matrix equation (1.6) of [WY] becomes:

$$\begin{pmatrix} -e & \frac{2}{c} \\ \frac{2}{c} & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \star \end{pmatrix}$$

which we solve in integers: $\lambda = \dfrac{2r}{c}$, $\bar{\lambda} = re$ for some $r \in \mathbb{Z}$ such that $2r$ is divisible by c. This solution yields horizontal surfaces H_1, H_2 as above such that the projection of H_1 to R_1 has degree $|\lambda|$ and that of H_2 to the base of M_2 has degree $|\bar{\lambda}|$ ([WY]). The boundary slopes of H_1, H_2 can be determined as follows. Assume $r \neq 0$ and recall $e \neq 0$. Then

$$\frac{\lambda}{\bar{\lambda}} = \frac{2}{ce}, \quad \frac{\bar{\lambda}}{\lambda} = \frac{ce}{2}.$$

Suppose that the slope of H_1 on T_j is given by $u_j \alpha_{1,j} + t_j \phi_{1,j}$ ($j = 1, 2$) and that of H_2 on T_j is $\bar{u}_j \bar{\alpha}_{2,j} + \bar{t}_j \bar{\phi}_{2,j}$ ($j = 1, 2, 3$). Then there are $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ so that equation (1.2) of [WY] takes on the form:

$$\frac{t_1}{u_1} = \frac{\epsilon_1 \bar{\lambda}}{-\lambda c} - \frac{d}{c} = \frac{-\epsilon_1 ce - 2d}{2c} \qquad\qquad \frac{\bar{t}_1}{\bar{u}_1} = \frac{\epsilon_1 \lambda}{-\bar{\lambda} c} + \frac{a}{-c} = \frac{-2\epsilon_1 - ace}{ec^2}$$

$$\frac{t_2}{u_2} = \frac{\epsilon_2 \bar{\lambda}}{\lambda c} - \frac{ce - d}{c} = \frac{2d - (2 - \epsilon_2)ce}{2c} \qquad\qquad \frac{\bar{t}_2}{\bar{u}_2} = \frac{\epsilon_2 \lambda}{\bar{\lambda} c} + \frac{a}{c} = \frac{2\epsilon_2 + ace}{ec^2}.$$

We require that $\frac{t_1}{u_1} + \frac{t_2}{u_2} = 0$ ([WY]), so $1 = \epsilon_2 = -\epsilon_1$. Hence

$$\frac{t_1}{u_1} = \frac{ce - 2d}{2c} \qquad\qquad \frac{\bar{t}_1}{\bar{u}_1} = \frac{2 - ace}{ec^2}$$

$$\frac{t_2}{u_2} = \frac{2d - ce}{2c} \qquad\qquad \frac{\bar{t}_2}{\bar{u}_2} = \frac{2 + ace}{ec^2}.$$

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Next we require $\frac{\bar{t}_1}{\bar{u}_1} + \frac{\bar{t}_2}{\bar{u}_2} + \frac{\bar{t}_3}{\bar{u}_3} = 0$, so

$$\frac{\bar{t}_3}{\bar{u}_3} = -[\frac{\bar{t}_1}{\bar{u}_1} + \frac{\bar{t}_2}{\bar{u}_2}] = -[\frac{2 - ace}{ec^2} + \frac{2 + ace}{ec^2}] = -\frac{4}{ec^2}.$$

Table 2 summarizes these calculations.

∂H_1	$i = 1$	$i = 2$
$\dfrac{t_i}{u_i}$	$\dfrac{ce - 2d}{2c}$	$\dfrac{2d - ce}{2c}$

∂H_2	$i = 1$	$i = 2$	$i = 3$
$\dfrac{\bar{t}_i}{\bar{u}_i}$	$\dfrac{2 - ace}{ec^2}$	$\dfrac{2 + ace}{ec^2}$	$-\dfrac{4}{ec^2}$

Table 2: The slopes of ∂H_1 and ∂H_2 when K has one component

Next recall that $c = 2$ and therefore

$$\bar{\lambda} = e, \quad \lambda = 1.$$

Further, by replacing B_0 and $\bar{B}_2$ by appropriate new sections of $Y_0 \to R_0$ and $M_2 \to \bar{R}_2$, we can take $b = 0$, $a = d = 1$. Then $\alpha_{2,1} = \bar{\alpha}_{2,1}$ and without loss of generality $B_2 = \bar{B}_2$. Since e is odd, these values determine $t_i, u_i, \bar{t}_i$, and $\bar{u}_i$. See Table 3.

∂H_1	$j = 1$	$j = 2$
t_j	$\dfrac{e - 1}{2}$	$\dfrac{1 - e}{2}$
u_j	1	1

∂H_2	$j = 1$	$j = 2$	$j = 3$
$\bar{t}_j$	$\dfrac{1 - e}{2}$	$\dfrac{1 + e}{2}$	-1
$\bar{u}_j$	e	e	e

Table 3: The coefficients of the slopes of ∂H_1 and ∂H_2 when K has one component

It follows from [WY], page 449, that $\partial H_1 \cap T_{1,i}$ has $|\frac{\lambda}{u_i}| = 1$ components $(i = 1, 2)$ and $\partial H_2 \cap T_{2,i}$ has $|\frac{\bar{\lambda}}{\bar{u}_i}| = 1$ components $(i = 1, 2, 3)$. Further, the fact that $\epsilon_1 = -\epsilon_2$ implies that any loop in H consisting of an arc between $T_{1,1}$ and $T_{1,2}$ on H_1 and an arc between $T_{2,2}$ and $T_{2,1}$ on H_2 is orientation reversing on H. Hence M is a surface semi-bundle. $\diamond$

Next we give concrete constructions for H_1 and H_2 and establish some of their specific properties for later use. We begin with H_2.

Recall that Y_2 is identified with $T \times [-\epsilon, \epsilon]$ for small $\epsilon > 0$, where $T = T \times \{0\}$, $T_{2,1} = T \times \{-\epsilon\}$ and $T_{2,2} = T \times \{\epsilon\}$. Set $\bar{\alpha}_0 = (\bar{B}_2 \cap T)$ and recall that it is a geodesic in T. We may suppose that $\bar{B}_2 = \bar{\alpha}_0 \times [-\epsilon, \epsilon]$. Orient $\bar{\alpha}_0$ to be homologous with $\bar{\alpha}_{2,1}$ in Y_2 and fix a new fibre $\bar{\phi}_0$ (with its given orientation) in T. Then T is identified with $\bar{\alpha}_0 \times \bar{\phi}_0$ and Y_2 with $\bar{\alpha}_0 \times \bar{\phi}_0 \times [-\epsilon, \epsilon]$. The first homology of each $\bar{\phi}$-vertical torus $T_* = \bar{\alpha}_0 \times \bar{\phi}_0 \times \{*\}$ has a canonical basis $\{\bar{\alpha}_*, \bar{\phi}_*\}$ where $\bar{\alpha}_* = \bar{B}_2 \cap T_*$ and $\bar{\phi}_* = (\bar{\phi}_0 \times [-\epsilon, \epsilon]) \cap T_*$ are oriented consistently with $\bar{\alpha}_0$ and $\bar{\phi}_0$. Note that the boundary slope of H_2 on $T_{-\epsilon} = T_{2,1}$ is $\dfrac{1}{2e} - \dfrac{1}{2}$ with respect to the ordered basis $\{\bar{\alpha}_{-\epsilon}, \bar{\phi}_{-\epsilon}\}$, while that on $T_\epsilon = T_{2,2}$ is $-\dfrac{1}{2e} - \dfrac{1}{2}$ with respect to $\{\bar{\alpha}_\epsilon, \bar{\phi}_\epsilon\}$ (c.f. the proof of Proposition 6.1 and Table 2.)

Decompose $\bar{\alpha}_0$ as the union of two intervals I, J which intersect along their boundaries. Let $0 < \delta < \epsilon$ and think of M_2 as

$$M_2 = (\bar{\alpha}_0 \times \bar{\phi}_0 \times [-\epsilon, \epsilon]) \setminus (\text{int}(I) \times \bar{\phi}_0 \times (-\delta, \delta)).$$

Fix a straight connected simple closed curve γ_1 in $T_{-\epsilon}$ of slope $\dfrac{1}{2e} - \dfrac{1}{2}$ with respect to the basis $(\bar{\alpha}_{-\epsilon}, \bar{\phi}_{-\epsilon})$. (Recall that an arc $\sigma \times \{*\} \subset T_*$ is *straight* if σ is geodesic in T and note that each new fibre of Y_2 is straight.) Set $\Theta_- = \gamma_1 \times [-\epsilon, -\delta]$. Then $\Theta_- \cap (J \times \bar{\phi}_0 \times \{-\delta\})$ consists of $|e|$ straight arcs of slope $\dfrac{1}{2e} - \dfrac{1}{2}$ in $T_{-\delta}$ evenly spaced around $\bar{\phi}$. Rotate these arcs about their centers as we pass from $-\delta$ to δ in $J \times \bar{\phi}_0 \times [-\delta, \delta]$, so that when we reach $J \times \bar{\phi}_0 \times \{\delta\}$, they have slope $-\dfrac{1}{2e} - \dfrac{1}{2}$. The trace of this rotation is a surface Θ_0 in $J \times \bar{\phi}_0 \times [-\delta, \delta]$ such that for each $t \in (-\delta, \delta)$, $\Theta_0 \cap T_t$ is a union of $|e|$ evenly spaced straight arcs of slope $-\dfrac{1}{2} - \dfrac{t}{2\delta e}$. When $t = \delta$, the arcs are contained in a straight loop of T_δ of slope $-\dfrac{1}{2e} - \dfrac{1}{2}$, which we denote by γ_2. Set $\Theta_+ = \gamma_2 \times [\delta, \epsilon]$. Finally define

$$H_2 = \Theta_- \cup \Theta_0 \cup \Theta_+.$$

The reader will verify that $H_2 \cap T_{2,3}$ has a single boundary component representing the slope $e\bar{\alpha}_{2,3} - \bar{\phi}_{2,3}$ and that H_2 is everywhere transverse to the new fibres. Thus we can form a surface fibration $\mathcal{F}_2$ in M_2 obtained by isotoping H_2 around the new fibres. Note also that H_2 is transverse to the original fibres except when $t = 0$. In this case $H_2 \cap (J \times \bar{\phi}_0 \times \{0\})$ consists of $|e|$ subarcs of the original fibres. Figure 6 illustrate one fibre (in red color) of $\mathcal{F}_2$ in M_2 when $e = 5$.

Next we construct H_1. Fix a properly embedded arc σ in B_1 running from $T_{1,1}$ to $T_{1,2}$ and let $\sigma \times [-1, 1]$ be a regular neighborhood of σ in B_1 such that $(\sigma \cap T_{1,1}) \times [-1, 1]$ is a
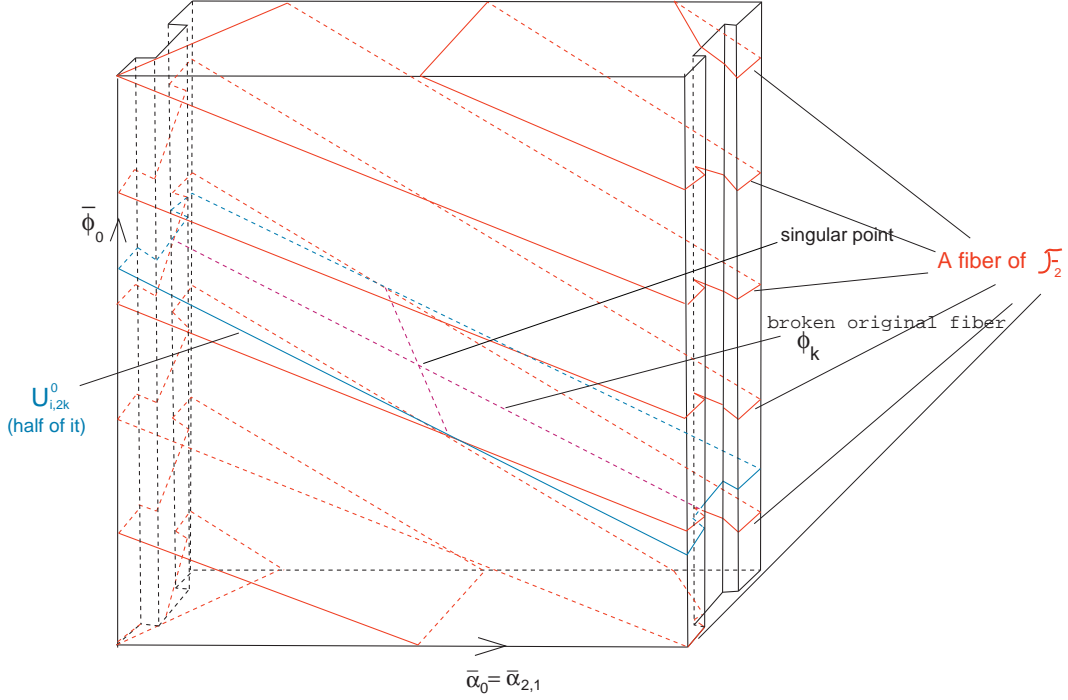
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Figure 6:

subarc of $\alpha_{1,1}$ which follows the orientation of $\alpha_{1,1}$ as we pass from -1 to 1 (and therefore $(\sigma \cap T_{1,2}) \times [-1,1]$ is a subarc of $\alpha_{1,2}$ which follows the opposite orientation of $\alpha_{1,2}$ as we pass from -1 to 1). The surface H_1 is obtained by replacing $\sigma \times [-1,1]$ in B_1 by a reimbedding of it which wraps around the ϕ-direction $(\frac{1-e}{2})$ times as we pass from -1 to 1 in $\sigma \times [-1,1] \times \phi$. By construction, H_1 is horizontal in Y_1 and one can easily verify that the boundary slopes of H_1 in $T_{1,1}$ and in $T_{1,2}$ are those given in Table 2. Let $\mathcal{F}_1$ be the corresponding surface bundle in Y_1.

Since H_2 is transverse to the original fibres in $T_{2,1} \cup T_{2,2}$, we may suppose that $\partial H_1 = \partial H_2$ in $T_1 \cup T_2$ and therefore produce a horizontal surface H in the graph manifold M. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ match up forming the corresponding global semi surface bundle of M, which we denote by $\mathcal{F}$.

Recall the torus $T = \hat{f}^{-1}(L_1^*)$ is vertical in the Seifert structure $\hat{f} : Y \to F$ and that $L_1 \subset T$ is geodesic in Y and transverse to the original fibres. Similarly each knot L_i $(i = 2, ..., p)$ is geodesic in Y, contained in the ϕ-vertical torus $U_i = \hat{f}^{-1}(L_i^*)$, and transverse to the original circle fibres.

For each i and $j \neq i$ we have $U_i \cap T = U_i \cap U_j = \phi_1 \cup ... \cup \phi_n$ where ϕ_k is the original

fibre $\hat{f}^{-1}(\hat{c}_k)$. The knot L_1 intersects each ϕ_k twice. Let U_i^0 be the $2n$-punctured torus $M \cap U_i$ $(i = 2, ..., p)$. It decomposes into $2n$ successive components $U_{i,m}^0$ $(m = 1, ..., 2n)$ where

$$\begin{cases} U_{i,2k-1}^0 \text{ is contained in } Y_1 \\ U_{i,2k}^0 \text{ is contained in } M_2 \end{cases}$$

$(k = 1, ..., n)$ and $U_{i,2k}^0 \cap \phi_k \neq \emptyset$ $(2 \leq i \leq p, 1 \leq k \leq n)$. Similarly the knot L_i decomposes into $4n$ successive segments $L_{i,m}$ $(2 \leq i \leq p, 1 \leq m \leq 4n)$ such that

$$\begin{cases} L_{i,2k-1} \cup L_{i,2k-1+2n} \text{ is contained in } U_{i,2k-1}^0 \\ L_{i,2k} \cup L_{i,2k+2n} \text{ is contained in } U_{i,2k}^0 \end{cases}$$

for $k = 1, ..., n$. Note that

$$L_{i,2k}^0 \cap \phi_k \neq \emptyset \qquad L_{i,2k+2n}^0 \cap \phi_k \neq \emptyset$$

$(2 \leq i \leq p, 1 \leq k \leq n)$. *Reorient L_i so that when $L_{i,m}$ is given the induced orientation,*

$$\text{the tail of } \begin{cases} L_{i,2k-1} \text{ lies in } T_2 \\ L_{i,2k} \text{ lies in } T_1 \end{cases}$$

for each $i = 2, ..., p$ and $k = 1, ..., 2n$. Finally, let $L_{i,m}^*$ be the image of $L_{i,m}$ in F $(2 \leq i \leq p, 1 \leq m \leq 2n)$.

Figure 6 illustrates how $U_{i,2k}^0$ intersects one leaf of $\mathcal{F}_2$ (where we assumed that the broken original fibre ϕ_k is contained the leave). From this it is not hard to see that $U_{i,2k}^0 \cap \mathcal{F}_2$ is as shown in Figure 7, up to a small isotopy in $U_{i,2k}$. Thus the induced singular foliation on $U_{i,2k}^0$ by $\mathcal{F}_2$ has only two singular points and they are contained in ϕ_k. Further, this singular foliation varies continuously under the isotopy of $U_{i,2k}^0$ to $U_{j,2k}^0$ which lies over a rotation of $L_{i,2k}^*$ to $L_{j,2k}^*$ centered at $\hat{c}_k$ which doesn't pass through L_1^*.

The following proposition will be needed later.

Proposition 6.2 *Fix a transverse orientation of $\mathcal{F}_2$. There is a smooth ϕ-vertical isotopy of the link $L_2 \cup \cdots \cup L_p$ in $\cup_{i=2}^p U_i^0$, fixed outside a small neighbourhood of $(\cup_{i=2}^p U_i^0) \cap M_2$, which repositions $L_{i,2k}$ to be transverse to $\mathcal{F}_2$ and to pass from the negative to the positive side of $\mathcal{F}_2$'s leaves while traveling from T_1 to T_2 in M_2 $(i = 2, ..., p, k = 1, ..., 2n)$ (c.f. Figure 7 for the position of $L_{i,2k} \cup L_{i,2k+2n}$ in $U_{i,2k}^0$ after the isotopy).*

Proof. Fix $i \in \{2, 3, ..., p\}$ and $k \in \{1, 2, ..., n\}$. As the geodesic arcs $L_{i,2k}, L_{i,2k+2n} \subset U_{i,2k}^0$ are perpendicular to the original fibres in $U_{i,2k}$, there is a smooth ϕ-vertical isotopy supported in $U_{i,2k}^0$ which places them in the positions depicted in Figure 7 near $U_{i,2k}^0 \cap (T \times$
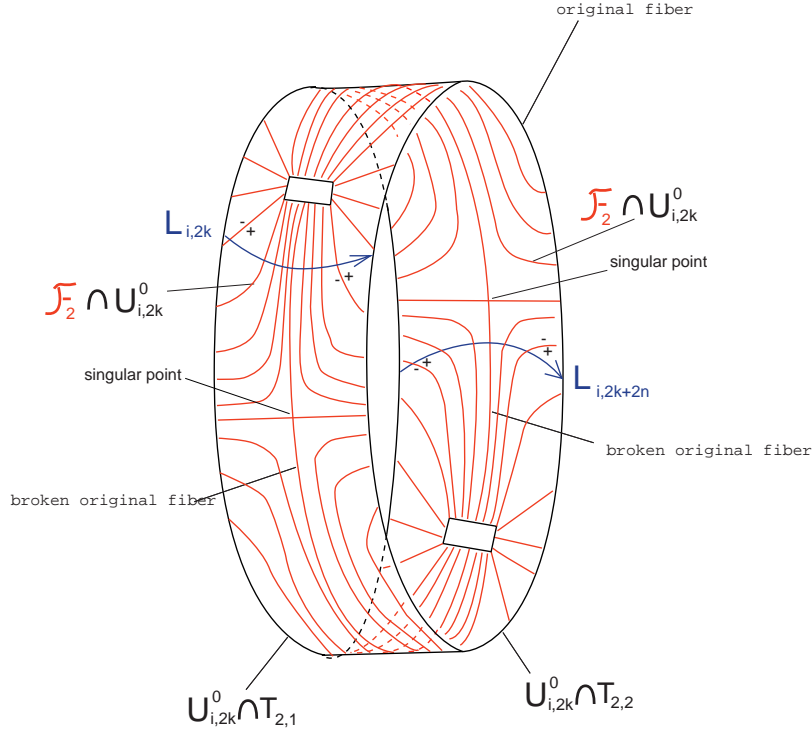
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Figure 7: $U_{i,2k}^0$

$[-\delta, \delta]$). A further ϕ-vertical isotopy, supported in a small neighbourhood of $U_{i,2k}^0$ in U_i^0, places them in the desired position throughout $U_{i,2k}^0$.

We would like to perform the isotopies of the previous paragraph simultaneously, but this is potentially obstructed since $U_{i,2k}^0 \cap U_{j,2l}^0 = \phi_k \cap M_2$ if $k = l$ (and is empty otherwise). In fact, there is no obstruction. To see this, observe that the broken fibre $U_{i,2k} \cap \phi_k$ consists of two arcs, each of which is split into two halves by a singular point of $U_{i,2k}^0 \cap \mathcal{F}_2$. These halves inherit a label "+" and "−" from the given orientation of ϕ_k. The reader will verify that for each k, the isotopy of the previous paragraph moves the finite set $(L_2 \cup \ldots \cup L_p) \cap \phi_k$ into either the +-sides or the −-sides of the arcs. It follows that the isotopies can be performed so as not to interfere with each other. This completes the proof. ◇

Let $p_2 : \breve{M} \to M$ be the double covering determined by the homomorphism

$$\pi_1(M) \to \mathbb{Z}/2, \delta \mapsto \overline{[\delta] \cdot [T]}$$

where $[\delta] \in H_1(M)$ is the class of δ and $[T] \in H_2(M)$ corresponds to a fundamental class of T. The reader will verify that $\breve{H} = p_2^{-1}(H)$ is a connected, orientable surface and the semifibre structure $\mathcal{F}$ on M lifts to a fibre structure $\breve{\mathcal{F}}$ on $\breve{M}$ with fibre $\breve{H}$. Note that

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$p_2^{-1}(Y_1), p_2^{-1}(M_2)$, and $p_2^{-1}(T_i)$ $(i = 1, 2)$ have two components each, say

$$p_2^{-1}(Y_1) = \breve{Y}_{1,1} \cup \breve{Y}_{1,2}$$

$$p_2^{-1}(M_2) = \breve{M}_{2,1} \cup \breve{M}_{2,2}$$

$$p_2^{-1}(T_i) = \breve{T}_{i,1} \cup \breve{T}_{i,2} \ (i = 1, 2)$$

Figure 8 illustrates the graph of the JSJ decomposition of $\breve{M}$. The cover $p_2 : \breve{M} \to M$ extends

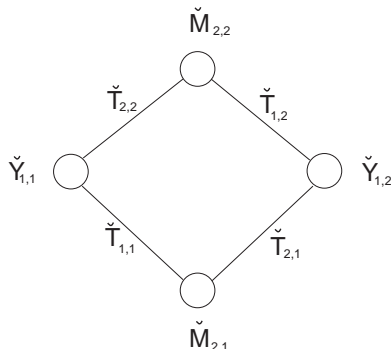Figure 8: The graph decomposition of $\breve{M}$.

to a (free) double cover $p_2 : \breve{Y} \to Y$ where $\breve{Y}$ inherits the structure of the total space of a locally trivial oriented circle bundle with fibre $\breve{\phi}$. Note as well that $\breve{Y}$ decomposes as

$$\breve{Y} = \breve{Y}_{2,1} \cup \breve{Y}_{1,1} \cup \breve{Y}_{2,2} \cup \breve{Y}_{1,2},$$

where $\breve{Y}_{2,j}$ $(j = 1, 2)$ are the two components of $p_2^{-1}(Y_2)$ with $\breve{M}_{2,j} \subset \breve{Y}_{2,j}$. Since L_i double covers L_i^* $(i = 1, ..., p)$, $p_2^{-1}(L_i)$ has two components which we denote by $\breve{L}_{i,d}$ $(d = 1, 2)$. Then $\breve{M}$ is the exterior of $\breve{L}_{1,1} \cup \breve{L}_{1,2}$ in $\breve{Y}$. Lemma 2.1 implies that wherever $\breve{L}_{i,d}$ meets a circle fibre of $\breve{Y}$, it meets it transversely.

Recall the surface bundles $\mathcal{F}_1, \mathcal{F}_2$ on Y_1, M_2 respectively, which match up to form the semi-bundle $\mathcal{F}$ in M. We denote by $\breve{\mathcal{F}}_{1,j}$ and $\breve{\mathcal{F}}_{2,j}$ the corresponding surface bundles in $\breve{Y}_{1,j}$ and $\breve{M}_{2,j}$ $(j = 1, 2)$ respectively. They match up in $\breve{M}$ to form the surface bundle $\breve{\mathcal{F}}$ in $\breve{M}$ described above. Fix a transverse orientation for $\breve{\mathcal{F}}$ and let $\breve{\mathcal{F}}_{i,j}$ $(i, j = 1, 2)$ have the induced transverse orientation. Thus each leaf of these foliations has a "$-$"-side and a "$+$"-side in such a way that the transverse orientation points from the former to the latter.

Set $\breve{U}_i = p_2^{-1}(U_i)$ (a $\breve{\phi}$-vertical torus) and $\breve{U}_i^0 = \breve{U}_i \cap \breve{M}$. Define $\breve{U}_{i,2k,j}^0 \subset \breve{M}_{2,j}$ to be the lift of $U_{i,2k} \subset M_2$, and $\breve{U}_{i,2k-1,j}^0 \subset \tilde{Y}_{1,j}$ that of $U_{i,2k-1}^0 \subset Y_1$. The intersection $\breve{U}_{i,2k,j}^0 \cap \breve{\mathcal{F}}_{2,j}$ is analogous to what is shown in Figure 7. Similarly define $\breve{L}_{i,2k,j} \cup \breve{L}_{i,2k+2n,j} \subset \breve{U}_{i,2k,j}^0$ to

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be the lift of $L_{i,2k} \cup L_{i,2k+2n} \subset U^0_{i,2k}$, and $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j} \subset \breve{U}^0_{i,2k-1,j}$ to be the lift of $L_{i,2k-1} \cup L_{i,2k+2n-1} \subset U^0_{i,2k-1}$. Endow $\breve{L}_{i,m,j}$ with the induced orientation.

By the nature of our constructions, the obvious analogue of Proposition 6.2 holds in each $\breve{M}_{2,j}$. In particular, there is a smooth $\breve{\phi}$-vertical isotopy of the link $\breve{L}_{2,1} \cup \breve{L}_{2,2} \cup \cdots \cup \breve{L}_{p,1} \cup \breve{L}_{p,2}$ in $\cup^p_{i=2} \breve{U}^0_i$, fixed outside a small neighbourhood of $(\cup^p_{i=2} \breve{U}^0_i) \cap (\breve{M}_{2,1} \cup \breve{M}_{2,2})$, which repositions $\breve{L}_{i,2k,j}$ to be transverse to $\breve{\mathcal{F}}_{2,j}$ and to pass from the negative to the positive side of $\breve{\mathcal{F}}_{2,j}$'s leaves while traveling from $\breve{T}_{1,j}$ to $\breve{T}_{2,j}$ in $\breve{M}_{2,j}$ ($i = 2, ..., p; k = 1, ..., 2n; j = 1, 2$).

For $j = 1, 2$, the surface bundle $\breve{\mathcal{F}}_{1,j}$ is transverse to the $\breve{\phi}$ Seifert structure on $\breve{Y}_{1,j}$. Hence it induces a fibring of $\breve{U}^0_{i,2k-1,j}$ by intervals transverse to the original circle foliation of $\breve{U}^0_{i,2k-1,j}$ ($i = 2, ..., p; k = 1, ..., n$). By construction, we may assume that near the boundary of $\breve{U}^0_{i,2k-1,j}$, the oriented arcs $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j} \subset \breve{U}^0_{i,2k-1,j}$ are transverse to the interval foliations and point from its negative to positive side (c.f. Figure 9 (a)). A typical interval fibre of $\breve{\mathcal{F}}_{1,j} \cap \breve{U}^0_{i,2k-1,j}$ is depicted in Figure 9 (a).
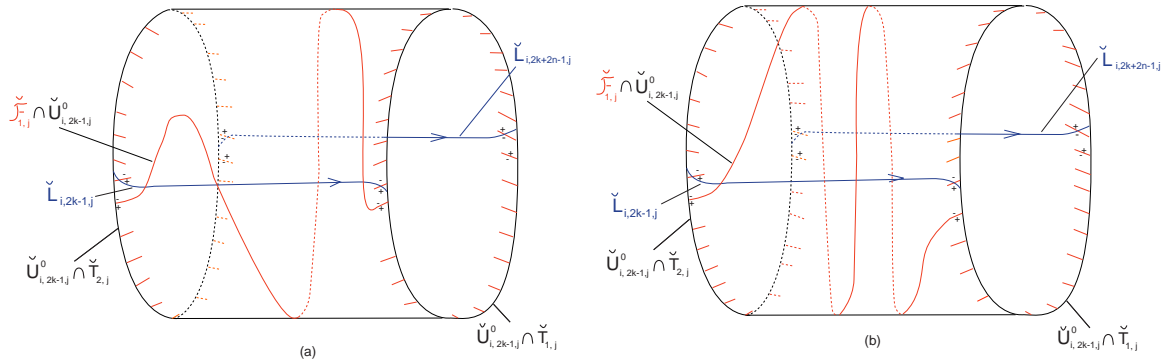

Figure 9: $\breve{U}^0_{i,2k-1,j}$

Note that both $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ are properly embedded arcs in $\breve{Y}_{1,j}$ joining the two boundary components of $\breve{Y}_{1,j}$. Hence if we take a $\breve{\phi}$-vertical torus $\breve{V}_j$ in int$(\breve{Y}_{1,j})$ which is parallel to one of the boundary components of $\breve{Y}_{1,j}$, then we may assume that $\breve{V}_j$ intersects each of $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ transversely in exactly one point. Our next adjustment is to perform a Dehn twist operation on $\breve{\mathcal{F}}_{1,j}$ in a neighborhood of $\breve{V}_j$ a sufficiently large number of times so that after a further isotopy, the resulting surface bundle in $\breve{Y}_{1,j}$ is transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$ everywhere ($i = 2, ..., p; k = 1, ..., n$). The details of this operation are as follows.

Let $N(\breve{V}_j) = \breve{V}_j \times I$ be a small neighborhood of $\breve{V}_j$ in int$(\breve{Y}_{1,j})$ consisting of $\breve{\phi}$-circle fibres. The boundary of $N(\breve{V}_j)$ consists of two $\breve{\phi}$-vertical tori, which we denote by $\breve{\partial}_1$ and $\breve{\partial}_2$. Note that $N(\breve{V}_j)$ intersects each of $\breve{L}_{i,2k-1,j}$ and $\breve{L}_{i,2k+2n-1,j}$ in a single arc. We may

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assume that the tail of the arc (with the induced orientation) is contained in $\breve{\partial}_1$ and its head in $\breve{\partial}_2$ $(i = 2, ..., p; k = 1, ..., n)$. Since $\breve{\mathcal{F}}_{1,j}$ is transverse to the old fibres, $\breve{\mathcal{F}}_{1,j} \cap N(\breve{V}_j)$ is a foliation of $N(\breve{V}_j)$ by annuli. We now perform a Dehn twist operation which wraps these annuli many times around the $\breve{\phi}$-fibres in the direction opposite to the transverse orientation of $\mathcal{F}_{1,j}$ as we pass from $\breve{\partial}_1$ to $\breve{\partial}_2$. This is illustrated Figure 10. Let $\breve{\mathcal{F}}'_{1,j}$ denote the resulting surface bundle in $\breve{Y}_{1,j}$.

Note that the foliation by intervals of $\breve{U}^0_{i,2k-1,j}$ determined by $\breve{U}^0_{i,2k-1,j} \cap \breve{\mathcal{F}}'_{1,j}$ is obtained from the foliation $\breve{U}^0_{i,2k-1,j} \cap \breve{\mathcal{F}}_{1,j}$ by the corresponding Dehn twists in a neighborhood of some $\breve{\phi}$-circle fibre in $\breve{U}^0_{i,2k-1,j}$ in the direction opposite the transverse orientation of $\breve{\mathcal{F}}_{1,j}$ as we pass from $\breve{\partial}_1 \cap \breve{U}^0_{i,2k-1,j}$ to $\breve{\partial}_2 \cap \breve{U}^0_{i,2k-1,j}$. Finally we adjust $\breve{\mathcal{F}}'_{1,j}$ by an isotopy which is the identity in a small regular neighborhood of $\partial \breve{Y}_{1,j}$ and outside a small regular neighborhood of $\cup_{i \geq 2,k,j} \breve{U}^0_{i,2k-1,j}$ so that the interval foliation in each $\breve{U}^0_{i,2k-1,j}$ becomes transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$. After the isotopy, a typical interval fibre in $\breve{U}^0_{i,2k-1,j}$ is depicted in Figure 9 (b). The resulting surface bundle, denoted $\breve{\mathcal{F}}''_{1,j}$, is everywhere transverse to $\breve{L}_{i,2k-1,j} \cup \breve{L}_{i,2k+2n-1,j}$ $(i = 2, ..., p; k = 1, ..., n)$. As this final operation is the identity in a neighborhood of a small regular neighborhood of $\partial \breve{Y}_{1,j}$, the resulting surface bundle $\breve{\mathcal{F}}''$ on $\breve{M}$ is transverse to the link $\cup_{i=2}^{p} \cup_{d=1}^{2} \breve{L}_{i,d}$ everywhere. The proof of Theorem 1.10 when K is a classic Montesinos knot is now complete.
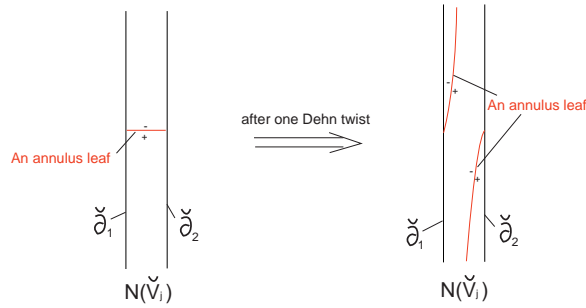

Figure 10: Dehn twist in $N(\breve{V}_j)$ in the direction opposite the transverse orientation of $\breve{\mathcal{F}}_{1,j}$.

6.2 K is a link of two components

The proof in this case is the same in most essentials and we continue to use the notation and conventions developed above except for the following changes.

In our present case, $\tilde{K} \to K^*$ is a trivial 2-fold cover and $\Psi^{-1}(\tilde{K})$ is a geodesic link L of $2p$ components $L_{1,1} \cup L_{1,2} \cup \ldots \cup L_{p,1} \cup L_{p,2}$ where $L_{i,1} \cup L_{i,2} = \hat{f}^{-1}(L_i^*)$. We take

M to be the exterior of $L_{1,1}$ in Y. As before, M admits a graph manifold decomposition $M = Y_1 \cup M_2$. Note that $L_{1,2}$ is a fibre of the Seifert structure on M_2.

Since K has two components and p is odd, $q_1 + \ldots + q_n$ is even (see §2). Note, moreover, that $0 \neq e(W_K) = -\frac{(q_1 + \ldots + q_n)}{p}$ so that $e = pe(W_K) = -(q_1 + \ldots + q_n)$ is even and non-zero. The numbers a, b, c, d from the proof of Proposition 6.1 can be taken to be $a = c = d = 1, b = 0$. In other words we have

$$\bar{\alpha} = \alpha$$

$$\bar{\phi} = \alpha + \phi$$

Our previous analysis can be easily modified to see that there are orientable horizontal surfaces H_1, H_2 in Y_1, M_2 which piece together to form the semifibre H where the projection of H_1 to R_1 has degree $|\lambda| = 1$ and that of H_2 to the base of M_2 has degree $|\bar{\lambda}| = \frac{e}{2}$. Further, if the slope of H_1 on T_j is given by $u_j \alpha_{1,j} + t_j \phi_{1,j}$ $(j = 1, 2)$ and that of H_2 on T_j is $\bar{u}_j \bar{\alpha}_{2,j} + \bar{t}_j \bar{\phi}_{2,j}$ $(j = 1, 2, 3)$ then reference to Table 2 shows that these slopes are given by

∂H_1	$i = 1$	$i = 2$
$\dfrac{t_i}{u_i}$	$\dfrac{e-2}{2}$	$\dfrac{2-e}{2}$

∂H_2	$i = 1$	$i = 2$	$i = 3$
$\dfrac{\bar{t}_i}{\bar{u}_i}$	$\dfrac{2-e}{e}$	$\dfrac{2+e}{e}$	$-\dfrac{4}{e}$

Table 4: The slopes of ∂H_1 and ∂H_2 when K has two components

Since e is even, the values of the coefficients $u_j, t_j, \bar{u}_j, \bar{t}_j$ are given in Table 5.

As before, we can think of H_2 as a surface which interpolates between the slope $-1 + \frac{2}{e}$ on T_1 and $-1 - \frac{2}{e}$ on T_2. The associated fibring $\mathcal{F}_2$ is transverse to all the new fibres, and in particular to $L_{1,2}$. The intersection of $U^0_{i,2k}$ with $\mathcal{F}_2$ is depicted in Figure 11. Our previous argument now proceeds without significant change to produce the desired surface bundle structure on $\check{M}_2$ which is transverse to $L_{1,2} \cup \ldots \cup L_{p,1} \cup L_{p,2}$. One point to note is that the ϕ-vertical isotopies of the geodesic arcs in $\check{M}_{2,j}$ can be made while isotoping $L_{1,2}$ through new fibres in T to a new position.

∂H_1	$j=1$	$j=2$
t_j	$\dfrac{e}{2}-1$	$1-\dfrac{e}{2}$
u_j	1	1

∂H_2	$j=1$	$j=2$	$j=3$ $e=4k+2$	$j=3$ $e=4k$
$\bar{t}_j$	$1-\dfrac{e}{2}$	$1+\dfrac{e}{2}$	-2	-1
$\bar{u}_j$	$\dfrac{e}{2}$	$\dfrac{e}{2}$	$\dfrac{e}{2}$	$\dfrac{e}{4}$

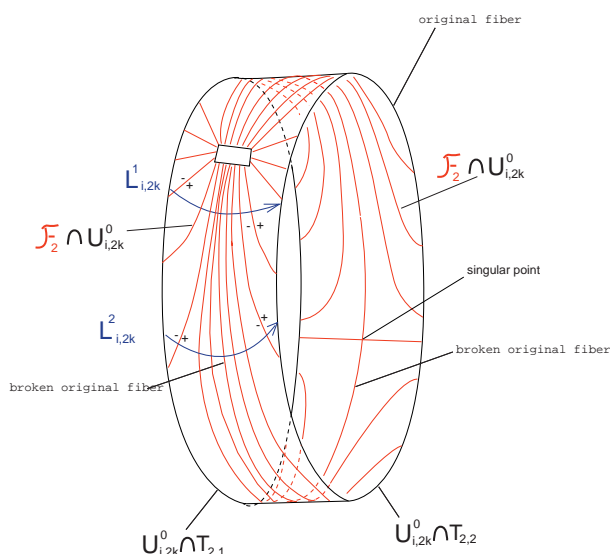Table 5: The coefficients of the slopes of ∂H_1 and ∂H_2 when K has two components

Figure 11: Dehn twist in $N(\check{V}_j)$ in the direction opposite the transverse orientation of $\check{\mathcal{F}}_{1,j}$.

7   Virtually bi-orderable fundamental groups

In this section we prove Theorem 1.11. The argument is based on the following result of Dale Rolfsen and Bernard Perron [PR].

Theorem 7.1 (Perron-Rolfsen) *If the monodromy of a surface bundle $M^3 \to S^1$ has positive eigenvalues on the level of first homology, then $\pi_1(M)$ is bi-orderable.* $\diamond$

It suffices then to show that for each of the fibre bundle structures associated to the generalised Montesinos links considered in this paper, some power of the monodromy has positive

eigenvalues. We begin with two elementary lemmas.

Lemma 7.2 *Let h_0 be a homeomorphism of a compact, connected, orientable surface F_0 and h the extension of h_0 to the surface F obtained by attaching 2-disks to ∂F_0. Then the set of eigenvalues of $(h_0)_* : H_1(F_0) \to H_1(F_0)$ consists of that of $h_* : H_1(F) \to H_1(F)$ together with some roots of unity.*

Proof. Since h_0 permutes the boundary components of F_0, the lemma's conclusion is a simple consequence of the exactness of the following commutative diagram's rows:

$$
\begin{array}{ccccc}
H_1(\partial F_0) & \xrightarrow{\ i_*\ } & H_1(F_0) & \xrightarrow{\ j_*\ } & H_1(F) \\
{\scriptstyle (h_0|\partial F_0)_*}\downarrow & & {\scriptstyle (h_0)_*}\downarrow & & {\scriptstyle h_*}\downarrow \\
H_1(\partial F_0) & \xrightarrow{\ i_*\ } & H_1(F_0) & \xrightarrow{\ j_*\ } & H_1(F)
\end{array}
$$

$\diamond$

Lemma 7.3 *If F is a compact, connected, orientable surface of genus 0 or 1, and $h : F \to F$ is an orientation preserving homeomorphism, then some power of $h_* : H_1(F) \to H_1(F)$ has positive eigenvalues. Hence any locally trivial F-bundle has a virtually bi-orderable fundamental group.*

Proof. The previous lemma yields the genus zero case immediately, while it reduces the genus 1 case to the situation where F is a torus and $h_* \in SL_2(\mathbb{Z})$. It is elementary to see that if the latter holds, the eigenvalues of h_* are real if $|\mathrm{trace}(h_*)| \geq 2$ and are roots of unity otherwise. Thus the lemma holds. $\diamond$

We observed in the introduction that generalised Montesinos links of spherical type virtually fibre with fibre a surface of genus 0, while the argument of §4.1 shows that the same is true for a generalised Montesinos links of type $S^2 \times \mathbb{R}$ unless it is a trivial link of two components. In the latter case, the link group is free on two generators, which is well known to be bi-orderable [MR]. Thus Theorem 1.11 holds in these cases. We noted in Remark 4.2 that generalised Montesinos links of type $\mathbb{E}^3$ or Nil virtually fibre with fibre a surface of genus 1, so the theorem follows in a similar fashion in these cases as well.

Suppose next that K has type $\mathbb{H}^2 \times \mathbb{R}$. Recall from §4.2 that there is a closed, connected, orientable, hyperbolic surface F and cover $\Psi : (Y, L) = (F \times S^1, \Psi^{-1}(\tilde{K})) \to (W_K, \tilde{K})$ where L_v is a union of circle fibres $\{x\} \times S^1$ and L_h is contained in a finite number of sections

$F \times \{u\}$. Lemma 3.2 shows that there is a finite degree cover $\psi : \tilde{F} \to F$ such that each component of $\tilde{L}_h = (\psi \times 1_{S^1})^{-1}(L_h) \subset \tilde{Y} = \tilde{F} \times S^1$ is homologically non-trivial in $\tilde{F}$. Take $\tilde{F}_0$ to be the exterior of $X = \tilde{L}_v \cap (\tilde{F} \times \{1\})$ in $\tilde{F} \times \{1\}$. The exterior of $\tilde{L}_v$ in $\tilde{Y}$ is the product $\tilde{Y}_0 = \tilde{F}_0 \times S^1$ and Theorem 1.7 implies that there is a surface bundle $p : \tilde{Y}_0 \to S^1$ whose fibres are everywhere transverse to $\tilde{L}_h$. Let S_0 denote one such fibre and let $h : S_0 \to S_0$ be the monodromy of p. It is not hard to see that $\chi(S_0) < 0$ and so as S_0 is essential in $\tilde{Y}_0 = \tilde{F}_0 \times S^1$, it can be isotoped to be horizontal with respect to the product Seifert structure. Hence h has finite order up to isotopy, so the eigenvalues of $h_* : H_1(S_0) \to H_1(S_0)$ are roots of unity. The exterior of $E(\tilde{L}_h)$ inherits a surface bundle structure whose monodromy g is determined as follows.

Identify $\tilde{F}_0 \times S^1$ cut open along $\tilde{F}_0 \times \{1\}$ with $\tilde{F}_0 \times I$ and think of $\tilde{L}_h$ as an isotopy of the subset $\tilde{L}_h \cap (\tilde{F}_0 \times \{0\})$ of $\tilde{F}_0$. This isotopy extends to an ambient isotopy $f_t : \tilde{F}_0 \to \tilde{F}_0$. Then $f_1(\tilde{F}_0) = \tilde{F}_0$ and $g = (h \circ f_1)|\tilde{F}_0$. Since $(f_1)_* = 1_{H_1(\tilde{F}_0)}$, Lemma 7.2 implies that the eigenvalues of g are roots of unity. Hence the Perron-Rolfsen theorem yields the desired conclusion.

Inspection of the construction in §4.4 shows that the argument of the previous paragraph can be applied to see that the fundamental groups of the exterior of a non-classic generalised Montesinos link of type $\widetilde{SL_2}$ are virtually bi-ordered.

Finally consider the case where K is one of the links $K(-g; e_0; \frac{q_1}{p}, \frac{q_2}{p}, ..., \frac{q_n}{p})$ considered in Theorem 1.10. From above we can suppose that K is a classic Montesinos link of type $\widetilde{SL_2}$. We'll describe the proof in the case where K is a knot. The case when it is a link is dealt with similarly.

First we suppose that $n = 0$. Then $e_0 = -e(W_K) \neq 0$ and so the exterior of K is Seifert fibred. Hence its fundamental group is virtually bi-ordered by [BRW]. Suppose then that $n > 0$. We constructed an $\check{H}$-bundle structure on the exterior $\check{M}$ of $\check{L}$ in $\check{Y}$ where $\check{H} = \check{H}_{1,1} \cup \check{H}_{2,1} \cup \check{H}_{1,2} \cup \check{H}_{2,2}$ where $H_{i,1} \cap H_{i,2} = \emptyset$ for $i = 1, 2$ and $H_{1,1} \cap H_{2,1} \cong H_{2,1} \cap H_{1,2} \cong H_{1,2} \cap H_{2,2} \cong H_{2,2} \cap H_{1,1} \cong S^1$. It follows that

$$H_1(\check{H}) \cong \mathbb{Z} \oplus H_1(\check{H}_{1,1}) \oplus H_1(\check{H}_{2,1}) \oplus H_1(\check{H}_{1,2}) \oplus H_1(\check{H}_{2,2})$$

By construction, the restriction of the monodromy h of the $\check{H}$-bundle exterior of $\check{L}$ to each $H_{i,j}$ is periodic. Thus there is some $m > 0$ such that $h_*^m|H_1(\check{H}_{i,j})$ is the identity and therefore has eigenvalues 1. Since $\det(h_*) = 1$, it follows that all the eigenvalues of h_*^m are 1. Since the virtual fibring of the exterior of K is obtained by removing a link (which is transverse to the fibres of the $\check{H}$-bundle structure) from the interior of $\check{M}$, Lemma 7.2 implies that the eigenvalues of the action of the monodromy on the first homology of the resulting surface fibre are roots of unity. Thus some power of it has eigenvalues 1. This

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observation is sufficient to complete the proof when K is a knot. ◇

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