Bulletin of the London Mathematical Society Advance Access published April 8, 2014

Bull. London Math. Soc. 1–14
doi:10.1112/blms/bdu002

Concordance of links with identical Alexander invariants

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Abstract

Davis showed that the topological concordance class of a link in the 3-sphere is uniquely determined by its Alexander polynomial for 2-component links with Alexander polynomial one. A similar result for knots with Alexander polynomial one was shown earlier by Freedman. We prove that these two cases are the only exceptional cases, by showing that the link concordance class is not determined by the Alexander invariants in any other case.

1. *Introduction*

Davis proved that if a 2-component link L has the Alexander polynomial of the Hopf link, namely $\Delta_L = 1$, then L is topologically concordant to the Hopf link [**14**]. In other words, for 2-components links, the topological concordance class is determined by the Alexander polynomial Δ_L when $\Delta_L = 1$. A natural question arises from this: *for which links does the Alexander polynomial determine the topological concordance class?*

The answer for knots is already known. A well-known result of Freedman (see [**17**], [**18**, 11.7B]) says that it holds for Alexander polynomial one knots, and Kim [**25**] (extending earlier work of Livingston [**28**]) showed that it does not hold for any Alexander polynomial that is not one.

The following main result of this note says that the results of Freedman and Davis are the only cases for which the topological link concordance class is determined by the Alexander polynomial.

Theorem A. *Suppose L is an m-component link, $m \geqslant 2$, and suppose $\Delta_L \neq 1$ if $m = 2$. Then there are infinitely many links $L = L_0, L_1, L_2, \ldots$ which have the same Alexander polynomial but are mutually not topologically concordant.*

Recall that the multivariable polynomial $\Delta_L = \Delta_L(x_1, \ldots, x_m)$ is well defined up to multiplication by $\pm x_1^{a_1} \ldots x_m^{a_m}$. In particular, $\Delta_L \neq 1$ means that Δ_L is not of the form $\pm x_1^{a_1} \ldots x_m^{a_m}$.

We remark that $\Delta_L \neq 1$ is automatically satisfied for $m \geqslant 3$, as a consequence of the Torres condition, which implies that $\Delta_L(1, \ldots, 1) = 0$ for $m \geqslant 3$. We also remark that in the smooth category, it is known that the conclusion of Theorem A holds even for knots and 2-component links with Alexander polynomial one. The knot case has been extensively studied in the literature; see, for example, [**16**, **21**] as early works. The case of links with unknotted components has been shown recently in [**4**].

Received 19 January 2013; revised 13 November 2013.

2010 *Mathematics Subject Classification* 57M25, 57M27, 57N70 (primary).

The first author was supported by NRF grants 2012-0009179 and 2013-053914 funded by the government of Korea. The second author thank the Indiana University Mathematics department, and especially Kent Orr, for its hospitality.

In fact, we can say more about the links L_i in Theorem A. To state the full result, we recall some terminologies in the following two paragraphs.

For an m-component link L in S^3, denote its exterior by $E_L = S^3 - \nu(L)$, where $\nu(L)$ denotes a tubular neighborhood of L. We always identify the boundary ∂E_L with $m(S^1 \times S^1)$ along the zero-framing, and view E_L as a bordered 3-manifold with this marking.

The notions of symmetric grope and Whitney tower concordance provide a framework for the study of link concordance. They measure the failure of links to be concordant in terms of fundamental geometric constructions, namely gropes and Whitney towers, in dimension 4. Roughly speaking, one defines a *height n (symmetric) Whitney tower concordance* by replacing the embedded annuli in the definition of concordance with transversely immersed annuli which form base surfaces supporting a Whitney tower of height n. A *height n (symmetric) grope concordance* is defined similarly by replacing annuli with disjointly embedded height n gropes. These were first used in the context of knot slicing by Cochran, Orr, and Teichner [9]. (Detailed definitions for arbitrary links can be found, for example, in [3, Section 2.4].) Also, in [3, Section 2.3], the first author introduced an analogue of these notions for bordered 3-manifolds, which is called an n-solvable cobordism. Roughly, an *n-solvable cobordism W* between bordered 3-manifolds M and M' is a 4-dimensional cobordism that induces $H_1(M) \cong H_1(W) \cong H_1(M')$ and admits a certain 'lagrangian' with 'duals' for the twisted intersection pairing on $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$, where $\pi = \pi_1(W)$ and $\pi^{(n)}$ is the nth derived subgroup (see Definition 2.3).

We can now state the full version of our main theorem.

Theorem B. *Suppose L_0 is an m-component link, and suppose $\Delta_L \neq 1$ if $m = 2$. Then there are infinitely many links $L_1, L_2, \dots$ satisfying the following conditions.*

(1) *For each i, there is a $\mathbb{Z}[\mathbb{Z}^m]$-homology equivalence of $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_{L_0}, \partial E_{L_0})$ rel ∂, namely $f|_\partial$ is the identification under the zero-framing and*

$$f_*\colon H_*(E_{L_i}; \mathbb{Z}[\mathbb{Z}^m]) \to H_*(E_{L_0}; \mathbb{Z}[\mathbb{Z}^m])$$

is an isomorphism.

(1′) *The following invariants are identical for all the L_i: Alexander polynomial, Alexander ideals, Blanchfield form [1], Milnor's $\bar{\mu}$-invariants [29], Orr's transfinite homotopy invariant θ_ω [30] (whenever defined), and Levine's homotopy invariant θ [27] (whenever defined).*

(2) *For any $i \neq j$, the exteriors E_{L_i} and E_{L_j} are not 2-solvably cobordant.*

(2′) *For any $i \neq j$, the links L_i and L_j are not height 4 grope concordant, not height 4 Whitney tower concordant, and not concordant.*

As references for the Alexander invariants appearing in Theorem B, see, for example, [23, 24].

Experts will easily see that Theorem B(1′) and (2′) are consequences of (1) and (2), respectively. In Section 2, we discuss this in more detail, including some background for the reader's convenience.

We remark that the links L_i in Theorem B can be chosen in such a way that they are indistinguishable to the eyes of the asymmetric Whitney tower/grope theory, which is another framework for the study of link concordance extensively investigated in recent work of Conant, Schneiderman, and Teichner (see [12] as an extended summary providing other references). Namely, the L_i (with the zero-framing) are mutually order n Whitney tower/grope concordant for any n in the sense of [13, Definition 3.1]. This is discussed in Section 5.

A key ingredient that we use to distinguish concordance classes of links is the Amenable Signature Theorem, which first appeared in [6]. It generalizes a result presented earlier in the influential work of Cochran–Orr–Teichner [9]. In [3], the first author formulated a symmetric

Whitney tower/grope framework for arbitrary links and bordered 3-manifolds, and gave (a refined version of) the Amenable Signature Theorem as a Cheeger–Gromov $\rho^{(2)}$-invariant obstruction to the existence of certain Whitney towers and gropes. In the proof of our main result, we use a special case of this, which is stated as Theorem 3.4 in this paper.

An interesting aspect of the proof of Theorem B is that it is separated into two cases which illustrate significantly the different aspects contained in the single problem, as discussed below.

For a 3-manifold M, the Cheeger–Gromov $\rho^{(2)}$-invariant $\rho^{(2)}(M,\phi)$ is a real number associated to a group homomorphism $\phi\colon \pi_1(M) \to \Gamma$, which we call a representation into Γ. We refer the reader to [**7**, **11**] for details. An essential requirement for the use of $\rho^{(2)}$-invariants in the study of concordance and related 4-dimensional equivalence relations is that the representation ϕ should have two properties: (1) ϕ does not annihilate certain interesting elements so that it does not lose too much information and (2) ϕ factors through the fundamental group of a relevant 4-manifold, for example, the exterior of a concordance, or more generally, a 4-manifold obtained by symmetric surgery on a Whitney tower or a grope.

To find such representations, first we consider links that are 'big' in the sense that they admit representations into non-abelian nilpotent quotients. It is straightforward to show (see Lemma 3.2) that a link L is 'big' if and only if L has either at least three components or if L is a 2-component link L with $\mathrm{lk}(L) \neq \pm 1$. For these links, we apply Dwyer's theorem to show that representations into certain nilpotent quotients have all the desired properties. (See Theorem 3.1.)

For links that are not big, we employ another approach using the *Blanchfield duality* of the link module $H_1(E_L;\mathbb{Z}[\mathbb{Z}^m])$. In fact, links that admit a nonzero Blanchfield pairing on the torsion part of the link module enable us to prove Theorem B using certain representations into *solvable* groups, which are not necessarily nilpotent. (See Theorem 4.1.) This applies especially to 2-component links with $\mathrm{lk}(L) \neq 0$ and $\Delta_L \neq 1$, which we may call 'small'. (See Lemma 4.2.) The case of small links resembles known approaches to the study of knot concordance [**2**, **9**] and it is related to earlier works of the authors [**3**, **20**].

The proofs of the nilpotent and solvable cases of Theorem B occupy Sections 3 and 4 respectively.

We remark that on their own, neither the class of 'big' links nor the class of 'small' links covers all the cases in Theorem B, while they have a significant overlap, for example 2-component links L with $|\,\mathrm{lk}(L)| > 1$. There are links that do not have useful nilpotent representations, for example 2-component links L with $|\,\mathrm{lk}(L)| = 1$, so that the Blanchfield pairing method is required as discussed above. On the other hand, there are links for which the Blanchfield pairing method fails to give any useful representations. An enlightening example is the Borromean rings. Its Alexander module is generated by the longitudes. Since the Blanchfield pairing automatically vanishes on the longitudes, it is apparent that the Blanchfield pairing cannot be used to prove property (1) for any representation ϕ.

Conventions. Manifolds are assumed to be topological and oriented, and submanifolds are assumed to be locally flat. When not specified, homology is with integer coefficients.

2. *Some observations on Theorem* B

In this section, we observe that Theorem B(1′) and (2′) are consequences of (1) and (2), respectively. We also discuss some necessary background.

Recall that a *link* L with m components in S^3 is a union of m disjoint oriented circles embedded in S^3. If $m = 1$, then it is called a *knot*. Two links L and L' are *concordant* if there is an h-cobordism (that is, disjoint union of annuli) between $L \times \{0\}$ and $L' \times \{1\}$ embedded in $S^3 \times [0,1]$.

2.1. *Alexander invariants and Blanchfield pairing*

We will first discuss the Alexander polynomial, Alexander ideals, and Blanchfield form. The *Alexander module* of a link L with m components is defined to be $H_1(E_L, \{*\}; \mathbb{Z}[\mathbb{Z}^m])$, viewed as a module over the group ring $\mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$, where the exterior E_L is endowed with the abelianization map $\pi_1(E_L) \to \mathbb{Z}^m$ (sending the i-meridian to the i-standard basis vector of $\mathbb{Z}^m$), and $*$ is a fixed basepoint in E_L. The module $H_1(E_L; \mathbb{Z}[\mathbb{Z}^m])$ is called the *link module* of L. The Alexander polynomial and Alexander ideals are determined by the Alexander module. It is also easy to see from the long exact sequence of a pair that the link module determines the Alexander module and vice versa. Therefore the conclusions in Theorem B(1′) on the Alexander polynomials and Alexander ideals are consequences of Theorem B(1).

Let $Q = \mathbb{Q}(x_1, \dots, x_m)$ be the quotient field of $\mathbb{Z}[\mathbb{Z}^m]$, namely the rational function field on m variables x_i. For a $\mathbb{Z}[\mathbb{Z}^m]$-module A, we denote its torsion part by

$$tA = \{x \in A \mid rx = 0 \text{ for some nonzero } r \in \mathbb{Z}[\mathbb{Z}^m]\}.$$

Owing to Blanchfield [1], there is a sesquilinear pairing

$$tH_1(E_L; \mathbb{Z}[\mathbb{Z}^m]) \times tH_1(E_L; \mathbb{Z}[\mathbb{Z}^m]) \to Q/\mathbb{Z}[\mathbb{Z}^m],$$

which is called the *Blanchfield pairing* of L. It is essentially defined by the duality of $(E_L, \partial E_L)$ over $\mathbb{Z}[\mathbb{Z}^m]$ coefficients. We also refer the reader to [23], particularly Section 2.3, for a thorough discussion of the Blanchfield pairing.

Since it is defined from duality, the Blanchfield pairing is functorial with respect to maps preserving the fundamental class, namely degree one maps on link exteriors. In particular, we have the following: the conclusion of Theorem B(1) that there is a $\mathbb{Z}[\mathbb{Z}^m]$-homology equivalence $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_{L_0}, \partial E_{L_0})$ implies that the Blanchfield pairings of L_i and L_0 are isomorphic. (Note that a $\mathbb{Z}[\mathbb{Z}^m]$-homology equivalence is automatically an integral homology equivalence and consequently a degree one map.)

2.2. *Milnor's invariants*

In [29], Milnor defined invariants $\bar{\mu}_L(I)$ for a link L with m components, where I is a finite sequence of integers in $\{1, \dots, m\}$. When I has length $|I|$, $\bar{\mu}_L(I)$ is called a $\bar{\mu}$-invariant of length $|I|$. This is the primary invariant for the study of structure peculiar to link concordance compared to the knot case.

Although $\bar{\mu}_L(I)$ is originally defined as a certain residue class of an integer, a known method to formulate that 'two links have the identical $\bar{\mu}$-invariants of length $\leqslant q$' in the strongest sense is as follows. Recall that the lower central series of a group π is defined by $\pi_1 := \pi$, $\pi_{q+1} = [\pi, \pi_q]$ where the bracket designates the commutator. We say that *two links L and L' with $\pi = \pi_1(E_L)$ and $G = \pi_1(E_{L'})$ have the same $\bar{\mu}$-invariants of length $\leqslant q$* if there is an isomorphism $h\colon \pi/\pi_q \to G/G_q$ that preserves (the conjugacy class) of each meridian and each 0-linking longitude.

Lemma 2.1. *Two links L and L' have the same $\bar{\mu}$-invariants of any length if there is an integral homology equivalence $f\colon (E_L, \partial E_L) \to (E_{L'}, \partial E_{L'})$ rel ∂.*

Proof. Let $\pi = \pi_1(E_{L_i})$ and $G = \pi_1(E_{L_0})$. By Stallings' theorem [33], f induces an isomorphism $h\colon \pi/\pi_q \cong G/G_q$. Since f is fixed on the boundary, h preserves the conjugacy classes of meridians and longitudes. □

Since the map f in the conclusion of Theorem B(1) is automatically an integral homology equivalence, it follows that the Milnor invariant conclusion in Theorem B(1′) is a consequence of Theorem B(1).

2.3. *Homotopy invariants of Orr and Levine*

In [30], Orr introduced a homotopy theoretic invariant of links which is still somewhat mysterious. For a link L, suppose all $\bar{\mu}$-invariants vanish. Then for a fixed homomorphism of the free group F on m generators into $\pi = \pi_1(E_L)$ that sends generators to meridians, we obtain an induced isomorphism $F/F_q \cong \pi/\pi_q$ by Stallings' theorem [33]. These give rise to $\pi \to \bar{F} := \varprojlim F/F_q$ and $E_L \to K(\bar{F}, 1)$. Let K_ω be the mapping cone of the map $K(F,1) \to K(\bar{F},1)$ induced by the inclusion $F \to \bar{F}$. Then it is easily seen that the map $E_L \to K(\bar{F},1) \to K_\omega$ extends to a map $o_L \colon S^3 \to K_\omega$. Its homotopy class $\theta_\omega(L) := [o_L] \in \pi_3(K_\omega)$ is Orr's transfinite homotopy invariant. It is unknown whether this invariant can be nonvanishing for links that have all $\bar{\mu}$-invariants zero.

Lemma 2.2. *If there is an integral homology equivalence $f \colon (E_L, \partial E_L) \to (E_{L'}, \partial E_{L'})$ rel ∂, then $\theta_\omega(L) = \theta_\omega(L')$.*

Proof. Let $\pi = \pi_1(E_L)$ and $G = \pi_1(E_{L'})$. Fix a map $\mu \colon F \to E_L$ sending generators to meridians. The map $f \circ \mu \colon F \to E_{L'}$ also sends generators to meridians. Define the map o_L and $o_{L'} \colon S^3 \to K_\omega$ as above, using μ and $f \circ \mu$. From the definition of o_L, it is easily seen that the map $g \colon S^3 \to S^3$ obtained by filling in f with the identity map of a solid torus satisfies $g \circ o_L = o_{L'}$. Since g has degree one, it follows that $\theta_\omega(L) = [o_L] = [o_{L'}] = \theta_\omega(L')$. □

This shows that the Orr invariant conclusion in Theorem B(1′) is a consequence of Theorem B(1). The same argument works for Levine's homotopy invariant $\theta(L)$ defined in [27]. We omit details.

2.4. *Solvable cobordism and Whitney tower/grope concordance*

The notion of an n-solvable cobordism used in Theorem B was formulated in [3, Section 2.3], as a version of an n-solution for manifolds with boundary. The notion of an n-solution was introduced by Cochran–Orr–Teichner [9], and was generalized by Harvey [22] to the case of links. For later use in this paper, we describe its precise definition below. Recall that a bordered 3-manifold M over a surface Σ is a 3-manifold with boundary identified with Σ, and for two bordered 3-manifolds M and M' over the same surface, a relative cobordism W is a 4-manifold satisfying $\partial W = M \cup_\partial -M'$.

Definition 2.3 (Solvable cobordism). We say that a relative cobordism W between bordered 3-manifolds M and M' is an *n-solvable cobordism* if (i) the inclusions induce $H_1(M) \cong H_1(W) \cong H_1(M')$ and (ii) there are homology classes $\ell_1, \dots, \ell_r, d_1, \dots, d_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$, where $\pi = \pi_1(W)$, whose images generate $H_2(W;\mathbb{Z})$, such that the $\mathbb{Z}[\pi/\pi^{(n)}]$-valued intersection pairing λ_n on $H_2(W;\mathbb{Z}[\pi/\pi^{(n)}])$ satisfies $\lambda_n(\ell_i, \ell_j) = 0$ and $\lambda_n(\ell_i, d_j) = \delta_{ij}$.

In [3, Section 2], the following was observed by using techniques in [9, Section 8]:

$$\begin{aligned} L \text{ and } L' \text{ are concordant} &\Longrightarrow L \text{ and } L' \text{ are height } n+2 \text{ grope concordant} \\ &\Longrightarrow L \text{ and } L' \text{ are height } n+2 \text{ Whitney tower concordant} \\ &\Longrightarrow E_L \text{ and } E_{L'} \text{ are } n\text{-solvably cobordant.} \end{aligned}$$

For the definitions of Whitney tower and grope concordance, refer the reader to, for example, [**3**, Definition 2.12, Definition 2.14].

From the above implications, we see that Theorem B(2′) is an immediate consequence of Theorem B(2).

3. *Links with nontrivial lower central series quotients*

The goal of this section is to prove the following special case of Theorem B. Recall that we denote the lower central series of a group π by $\{\pi_q\}$.

THEOREM 3.1. *Suppose L is an m-component link with $\pi = \pi_1(E_L)$ such that $\pi_2/\pi_3 \neq 0$. Then there are infinitely many links $L = L_0, L_1, L_2, \dots$ such that there is a $\mathbb{Z}[\mathbb{Z}^m]$-homology equivalence $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_{L_0}, \partial E_{L_0})$ rel ∂ for each i but the exteriors E_{L_i} and E_{L_j} are not 2-solvably cobordant for any $i \neq j$.*

From Theorem 3.1 and the discussions in Section 2, it follows that Theorem B holds whenever $\pi_2/\pi_3 \neq 0$.

Before we prove Theorem 3.1, we clarify when the lower central series hypothesis is satisfied.

LEMMA 3.2. *Suppose L is an m-component link with $\pi = \pi_1(E_L)$, $m \geqslant 2$.*

(1) *The abelian group π_2/π_3 has rank $\geqslant (m-1)(m-2)/2$.*
(2) *If $m = 2$, then $\pi_2/\pi_3 \cong \mathbb{Z}/\operatorname{lk}(L)\mathbb{Z}$.*

Consequently, $\pi_2/\pi_3 \neq 0$ if and only if either (i) *$m \geqslant 3$ or* (ii) *$m = 2$ and $\operatorname{lk}(L) \neq \pm 1$.*

Proof. Milnor [**29**, Theorem 4] showed that π/π_3 is presented by

$$\pi/\pi_3 = \langle x_1, \dots, x_m \mid [x_1, \lambda_1], \dots, [x_m, \lambda_m], F_3 \rangle,$$

where m is the number of components of L, F_3 is the third lower central subgroup of the free group F on $x_1, \dots, x_m$, and λ_i is an element in F which represents the ith longitude of L in π/π_3. It is well known that F_2/F_3 is the free abelian group generated by the basic commutators $[x_i, x_j]$, $i < j$, by Hall's basis theorem. Also, we have $\lambda_i \equiv \prod_{j \neq i} x_j^{\ell_{ij}} \bmod F_2$, where ℓ_{ij} is the linking number of the ith and jth components of L. Using the standard identities $[a, bc] \equiv [a, b][a, c] \bmod F_3$ and $[a, b]^{-1} = [b, a]$, we obtain

$$[x_i, \lambda_i] \equiv [x_1, x_i]^{-\ell_{1i}} \dots [x_{i-1}, x_i]^{-\ell_{(i-1)i}} [x_i, x_{i+1}]^{\ell_{i(i+1)}} \dots [x_i, x_m]^{\ell_{im}} \bmod F_3.$$

From this it follows that π_2/π_3 is given by the abelian group presentation with $m(m-1)/2$ generators $v_{ij} = [x_i, x_j]$, $1 \leqslant i < j \leqslant m$, and the following m relators for $i = 1, \dots, m$:

$$-\ell_{i1} v_{1i} - \dots - \ell_{(i-1)i} v_{(i-1)i} + \ell_{i(i+1)} v_{i(i+1)} + \dots + \ell_{im} v_{im} = 0.$$

Note that the m relators add up to zero. Therefore, the rank of π_2/π_3 is at least $m(m-1)/2 - (m-1) = (m-1)(m-2)/2$.

For $m = 2$, then we have one generator v_{12} and one relator $\ell_{12} v_{12} = 0$. Therefore $\pi_2/\pi_3 \cong \mathbb{Z}/\ell_{12}\mathbb{Z}$. □

In the proof of Theorem 3.1, we will make use of the following definition.

Definition 3.3. For a group G and a sequence $\mathcal{P} = (R_1, R_2, \ldots)$ of commutative rings R_i with unity, we define the *mixed-coefficient lower central series* $\{\mathcal{P}_q G\}$ by $\mathcal{P}_1 G := G$ and

$$\mathcal{P}_{q+1} G := \operatorname{Ker}\left\{\mathcal{P}_q G \to \frac{\mathcal{P}_q G}{[G, \mathcal{P}_q G]} \underset{\mathbb{Z}}{\otimes} R_q\right\}.$$

We remark that $\mathcal{P}_q G$ is a characteristic, and therefore normal, subgroup of G. We will also make use of the following result from [3]:

Theorem 3.4 (A special case of the Amenable Signature Theorem [3, Theorem 3.2]). *Suppose W is a 2-solvable cobordism between bordered 3-manifolds M and M'. Suppose Γ is a group which admits a filtration $\{e\} \subset \Gamma' \subset \Gamma$ such that Γ/Γ' is torsion-free abelian and such that Γ' is either torsion-free abelian or an abelian p-group for some prime p. Then for any $\phi\colon \pi_1(M \cup_\partial -M') \to \Gamma$ that extends to $\pi_1(W)$, $\rho^{(2)}(M \cup_\partial -M', \phi) = 0$.*

Proof. First note that Γ is a solvable group and therefore amenable. Furthermore, it follows from [6, Lemma 6.8] that Γ lies in Strebel's class $D(R)$ [34] for $R = \mathbb{Q}$ or $R = \mathbb{Z}_p$, with p a prime. The theorem is now an immediate consequence of case III of the Amenable Signature Theorem [3, Theorem 3.2], since $\Gamma^{(2)} = \{e\}$ and W is a 2-solvable cobordism. □

We are now ready to give the proof of Theorem 3.1. If the reader is interested in link concordance only, then in the proof below the phrase '2-solvable cobordism' can be safely replaced with 'concordance exterior'.

Proof of Theorem 3.1. By the hypothesis, there exists a simple closed curve α in E_L, which is a generator of the abelian group π_2/π_3. We can and will assume that α is unknotted in S^3. We then choose a prime p which divides the order of α in π_2/π_3; if α has infinite order, choose any prime p.

Let $\mathcal{P} = (\mathbb{Q}, \mathbb{Z}_p)$, so that $\mathcal{P}_q G$ is defined for $q = 1, 2, 3$. Then, for not only the given link group $\pi = \pi_1(E_L)$ but also any π with $\pi/[\pi, \pi]$ torsion free, $\mathcal{P}_2\pi$ is the ordinary lower central subgroup $\pi_2 = [\pi, \pi]$. Also, $\mathcal{P}_2\pi/\mathcal{P}_3\pi \cong (\pi_2/\pi_3) \otimes \mathbb{Z}_p$, a $\mathbb{Z}_p$-vector space. Consequently, for the given $\pi = \pi_1(E_L)$, our α represents a nonzero element in $\mathcal{P}_2\pi/\mathcal{P}_3\pi \subset \pi/\mathcal{P}_3\pi$, namely an element of order p.

According to Cheeger and Gromov [7, p. 23] there is a constant $R > 0$ determined by the 3-manifold $E_L \cup_\partial -E_L$ such that $|\rho^{(2)}(E_L \cup_\partial -E_L, \Phi)| < R$ for any homomorphism Φ. Let us choose knots J_i inductively for $i = 1, 2, \ldots$ in such a way that the inequality

$$|\rho^{(2)}(J_i, \mathbb{Z}_p)| > R + |\rho^{(2)}(J_j, \mathbb{Z}_p)| \tag{1}$$

is satisfied whenever $i > j$. Here, given a knot J and $p \in \mathbb{Z}$, we write

$$\rho^{(2)}(J, \mathbb{Z}_p) := \rho^{(2)}(\text{0-framed surgery on } J, \text{unique epimorphism onto } \mathbb{Z}_p).$$

For example, the connected sum of a sufficiently large number of trefoils can be taken as J_i. To see this, denote the right-handed trefoil by T and set $C := (1/p)\sum_{r=0}^{p-1} \sigma_T(e^{2\pi r\sqrt{-1}/p})$, where $\sigma_T(z)$ denotes the Levine–Tristram signature of T corresponding to $z \in S^1$. It is straightforward to see that $C > 0$, since at least one of the $\sigma_T(z)$ is positive, and all nonzero $\sigma_T(z)$ have the same sign. One can compute the signature of the hermitian matrix $A := (1-\omega)V + (1-\bar{\omega})V^T)$, where $V := \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is a Seifert matrix for the trefoil, for $\omega \in S^1 \setminus \{1\} \subset \mathbb{C}$, to be either 2 or 0, with the latter occurring for $\operatorname{Re}(\omega) > 1/2$. If J is the connected sum of k copies of T, then we

see that

$$\rho^{(2)}(J,\mathbb{Z}_p) = \frac{1}{p}\sum_{r=0}^{p-1}\sigma_J(e^{2\pi r\sqrt{-1}/p}) = k\cdot C. \tag{2}$$

Here, for the first equality we appeal to [**19**, Corollary 4.3] or equivalently [**6**, Lemma 8.7(2)] and for the second equality we use the additivity of the Levine–Tristram signatures. If we denote the connected sum of $i\cdot\lceil R/C\rceil$ copies of T by J_i, then (1) is satisfied.

We then use the satellite construction to produce a new link $L_i := L(\alpha, J_i)$ by tying the knot J_i into L along the curve α. More precisely, by filling in the exterior $E_\alpha = S^3 - \nu(\alpha)$ with the exterior $E_{J_i} = S^3 - \nu(J_i)$ along an orientation-reversing homeomorphism of the boundary torus $\partial\nu(\alpha) \to \partial\nu(J_i)$ that identifies a meridian and 0-linking longitude of α with a 0-linking longitude and a meridian of J_i, respectively, we obtain a new 3-manifold which is homeomorphic to S^3, and the image of $L \subset E_\alpha$ under this homeomorphism is the new link $L_i = L(\alpha, J_i)$. We denote a 0-framed push-off of α in $E_{L\cup\alpha} \subset E_{L_i}$ by α_i.

It is well known that there is an integral homology equivalence $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_L, \partial E_L)$ (see, for example, [**5**, Lemma 5.3]). In fact, f is obtained by gluing the identity map on $E_{L\cup\alpha}$ with the standard homology equivalence $(E_{J_i}, \partial E_{J_i}) \to (E_{J_0}, \partial E_{J_0}) = S^1 \times (D^2, S^1)$. Since α lies in $[\pi_1(E_L), \pi_1(E_L)]$, a Mayer–Vietoris argument applied to the above construction shows that f induces isomorphisms on $H_*(-;\mathbb{Z}[\mathbb{Z}^m])$.

By Stallings' theorem [**33**] and our above discussion on $\mathcal{P}_q\pi/\mathcal{P}_{q+1}\pi$ for $q<3$, we have an induced isomorphism $\pi_1(E_{L_i})/\mathcal{P}_3\pi_1(E_{L_i}) \cong \pi/\mathcal{P}_3\pi$. Since f restricts to the identity on $E_{L\cup\alpha}$, the element α_i corresponds to α under this isomorphism.

We will need the following lemma, which is a consequence of Dwyer's Theorem [**15**], a generalization of Stallings' theorem. We remark that for the special case of a concordance exterior, Stallings' theorem can be used instead.

Lemma 3.5. *If W is a 1-solvable cobordism between two bordered 3-manifolds M and M' with torsion-free $H_1(M)$, then the inclusions induce isomorphisms*

$$\pi_1(M)/\mathcal{P}_3\pi_1(M) \cong \pi_1(W)/\mathcal{P}_3\pi_1(W) \cong \pi_1(M')/\mathcal{P}_3\pi_1(M').$$

Proof. Recall Dwyer's theorem [**15**]: if $f\colon X \to Y$ induces an isomorphism $H_1(X) \cong H_1(Y)$ and an epimorphism

$$H_2(X) \to H_2(Y)/\operatorname{Im}\{H_2(Y;\mathbb{Z}[\pi_1(Y)/\pi_1(Y)_q]) \longrightarrow H_2(Y)\},$$

then f induces an isomorphism $\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(Y)_q/\pi_1(Y)_{q+1}$.

We want to apply this twice with $Y = W$ both times, and with $X = M$ and then $X = M'$. By the definition of an n-solvable cobordism, we have $H_1(M) \cong H_1(W) \cong H_1(M')$. Also, there are 1-lagrangian elements $\ell_1, \dots, \ell_r$ with 1-duals $d_1, \dots, d_r$ lying in $H_2(W;\mathbb{Z}[\pi_1(W)/\pi_1(W)^{(1)}])$ such that the ℓ_i and d_j generate $H_2(W)$. Since $\pi_1(W)^{(1)}$ is equal to $\pi_1(W)_2$, the H_2 condition of Dwyer's theorem is satisfied. Therefore, it follows that

$$\pi_1(M)_q/\pi_1(M)_{q+1} \cong \pi_1(W)_q/\pi_1(W)_{q+1} \cong \pi_1(M')_q/\pi_1(M')_{q+1}$$

for $q = 1, 2$ by Dwyer's theorem. By our observation that $\mathcal{P}_2\pi = \pi_2$ and $\mathcal{P}_2\pi/\mathcal{P}_3\pi = \pi_2/\pi_3 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for groups π with torsion-free H_1, we obtain

$$\mathcal{P}_q\pi_1(M)/\mathcal{P}_{q+1}\pi_1(M) \cong \mathcal{P}_q\pi_1(W)/\mathcal{P}_{q+1}\pi_1(W) \cong \mathcal{P}_q\pi_1(M')/\mathcal{P}_{q+1}\pi_1(M')$$

for $q = 1, 2$. From this the desired conclusion follows by the five lemma. □

Returning to the proof of Theorem 3.1 let W be a 2-solvable cobordism between E_{L_i} and E_{L_j}. We will show that $i = j$. First note that we obtain

$$\pi/\mathcal{P}_3\pi \cong \pi_1(E_{L_i})/\mathcal{P}_3\pi_1(E_{L_i}) \cong \pi_1(W)/\mathcal{P}_3\pi_1(W) \cong \pi_1(E_{L_j})/\mathcal{P}_3\pi_1(E_{L_j})$$

by Lemma 3.5. Let $\phi\colon \pi_1(W) \to \Gamma := \pi_1(W)/\mathcal{P}_3\pi_1(W)$ be the projection, and by abuse of notation, we denote its restriction to $\partial W = E_{L_i} \cup_\partial -E_{L_j}$ by ϕ as well. Since α represents an order p element in $\pi/\mathcal{P}_3\pi$, both $\phi([\alpha_i])$ and $\phi([\alpha_j])$ have order p.

By applying Theorem 3.4 with $n = 2$, we obtain that

$$\rho^{(2)}(E_{L_i} \cup_\partial -E_{L_j}, \phi) = 0. \tag{3}$$

On the other hand, note that the map ϕ induces a homomorphism $\varphi\colon \pi_1(E_L \cup_\partial -E_L) \to \Gamma$ as follows. Recall that $E_{L_i} \cup_\partial -E_{L_j}$ is obtained from $E_L \cup_\partial -E_L$ by satellite constructions using the knots J_i and J_j. Viewing E_{J_i} as a subspace of $E_{L_i} \cup_\partial -E_{L_j}$, the homomorphism ϕ restricted to $\pi_1(E_{J_i})$ sends the meridian of J_i to $\phi([\alpha_i])$. Since $\phi([\alpha_i])$ has order p in the abelian subgroup $\mathcal{P}_2\pi/\mathcal{P}_3\pi$ of Γ, it follows that ϕ restricted to E_{J_i} factors as $\pi_1(E_{J_i}) \to \mathbb{Z} \twoheadrightarrow \mathbb{Z}_p \hookrightarrow \Gamma$ where the first map is the abelianization. Similarly for J_j. It follows that ϕ on $\pi_1(E_{L_i} \cup_\partial -E_{L_j})$ gives rise to a homomorphism $\varphi\colon \pi_1(E_L \cup_\partial -E_L) \to \Gamma$. To see this, observe that we can arrange an element $\gamma \in \pi_1(E_L \cup_\partial E_L)$ to avoid $\nu(\alpha_i)$ and $\nu(\alpha_j)$. The image $\varphi(\gamma)$ can then be defined by ϕ. This is well defined because crossing α_i in homotopy of γ changes γ by a meridian of $\nu(\alpha_i)$, which in E_{L_i} is attached to a longitude of J_i, and therefore maps trivially under ϕ.

Now, using (i) the additivity of $\rho^{(2)}$ under satellite construction [**10**, Proposition 3.2], (ii) the L^2-induction property of $\rho^{(2)}$ [**9**, Proposition 5.13], and (iii) the fact that α represents an element of order p in Γ, we obtain that

$$\rho^{(2)}(E_{L_i} \cup_\partial -E_{L_j}, \phi) = \rho^{(2)}(E_L \cup_\partial -E_L, \varphi) + \rho^{(2)}(J_i, \mathbb{Z}_p) - \rho^{(2)}(J_j, \mathbb{Z}_p). \tag{4}$$

Fact (i) is proved using a cobordism, over which ϕ extends, from $E_{L_i} \cup_\partial -E_{L_j}$ to the disjoint union of $E_L \cup_\partial -E_L$, M_{J_i} and $-M_{J_j}$, where M_K is the zero-framed surgery manifold of a knot K, together with the fact that $\rho^{(2)}$ invariants are additive under disjoint union. Fact (iii) tells us that, for $k = i, j$, the restriction of ϕ to M_{J_k} factors through a homomorphism $\pi_1(M_{J_k}) \to \mathbb{Z}_p$, which itself factors through the abelianization. Fact (ii) then implies that we may equate the $\rho^{(2)}$ invariant of M_{J_k} with $\rho^{(2)}(J_i, \mathbb{Z}_p)$, as desired.

It now follows from (3) and the choice of R that

$$|\,|\rho^{(2)}(J_i, \mathbb{Z}_p)| - |\rho^{(2)}(J_j, \mathbb{Z}_p)|\,| < R.$$

In light of (1) we now see that $i = j$.

Thus, when $i \neq j$, we have shown that the bordered manifolds E_{L_i} and E_{L_j} are not 2-solvably cobordant. □

4. *Links with nontrivial Blanchfield pairing*

According to Lemma 3.2 the fundamental group of a 2-component link with linking number equal to ± 1 admits no non-abelian nilpotent quotients. The goal of this section is to provide an alternative approach, using the Blanchfield duality, to prove Theorem B for such links.

In this section, we denote $\mathbb{Q}[\mathbb{Z}^m]$ by Λ, where m is understood to be the number of components of the link. The ring $\mathbb{Q}[\mathbb{Z}^m]$ has the same quotient field as $\mathbb{Z}[\mathbb{Z}^m]$, namely the rational function field $Q = \mathbb{Q}(x_1, \ldots, x_m)$ considered in Section 2. Recall that for a Λ-module A, we denote the Λ-torsion submodule by tA. In what follows, $B\ell_L$ denotes the *rational* Blanchfield form $B\ell_L\colon tH_1(E_L;\Lambda) \times tH_1(E_L;\Lambda) \to Q/\Lambda$. We remark that this Blanchfield pairing is obtained by tensoring the integral Blanchfield pairing discussed in Section 2 with $\mathbb{Q}$.

THEOREM 4.1. *Suppose L is an m-component link for which the Blanchfield pairing $B\ell_L$ is not constantly zero, that is, $B\ell_L(x,y) \neq 0$ for some $x, y \in tH_1(E_L;\Lambda)$. Then there are infinitely many links $L = L_0, L_1, L_2, \dots$ such that there is a $\mathbb{Z}[\mathbb{Z}^m]$-homology equivalence of $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_{L_0}, \partial E_{L_0})$ rel ∂ for each i but the exteriors E_{L_i} and E_{L_j} are not 2-solvably cobordant (and consequently the links L_i and L_j are not concordant) for any $i \neq j$.*

Before proving Theorem 4.1, we observe a special case to which Theorem 4.1 applies.

LEMMA 4.2. *Suppose L is a 2-component link with $\operatorname{lk}(L) \neq 0$ and $\Delta_L \neq 1$. Then the Blanchfield pairing $B\ell_L$ is not constantly zero.*

Proof. Since $\operatorname{lk}(L) \neq 0$, the Blanchfield pairing $B\ell_L$ on $H_1(E_L;\Lambda)$ is nondegenerate by Levine [**26**, Theorem B]. Therefore, it suffices to show that $H_1(E_L;\Lambda)$ is nonzero.

Recall that the Torres condition (see, for example, [**23**, Section 5.1]) implies that, up to multiplication by a monomial, the following equality holds:

$$\Delta_L(x_1, 1) = (x_1^{|\operatorname{lk}(L)|-1} + \dots + x_1 + 1)\Delta_K(x_1).$$

Here, K is the first component of L. In particular, we see that $\Delta_L(1,1) = |\operatorname{lk}(L)|$. Our assumptions that $\operatorname{lk}(L) \neq 0$ and $\Delta_L \neq 1$ now immediately imply that Δ_L is not a monomial. It follows that $H_1(E_L;\Lambda)$ is nonzero. □

From Theorem 4.1, Lemma 4.2 and the discussions in Section 2, it follows that Theorem B holds for 2-component links with nonzero linking number and with $\Delta_L \neq 1$. This, combined with Theorem 3.1 and Lemma 3.2, completes the proof of Theorem B, modulo the proof of Theorem 4.1 which is given below.

Proof of Theorem 4.1. We choose simple closed curves $\alpha_1, \dots, \alpha_N$ in E_L which are unknotted in S^3 and have linking number zero with L such that their classes $[\alpha_k]$ generate $tH_1(E_L;\Lambda)$, which is a finitely generated module since Λ is Noetherian. For each $i = 1, 2, \dots$, we use the satellite construction to produce a new link $L_i = L(\{\alpha_k\}, \{J_{ik}\})$ by tying a collection of knots $\{J_{ik}\}_{k=1}^N$ into L along the curves α_k, for $k = 1, \dots, N$. (See the proof of Theorem 3.1 for a more detailed description of the satellite construction.) We define J_{0k} to be the trivial knot for each k, so that $L = L_0 = L(\{\alpha_k\}, \{J_{0k}\})$ is also described in the same way. For $i \geqslant 1$, we choose the knots J_{ik} as follows. As made use of in the proof of Theorem 3.1, due to Cheeger–Gromov [**7**, p. 23], there is a constant $C > 0$ determined by the 3-manifold $E_L \cup_\partial -E_L$ such that $|\rho^{(2)}(E_L \cup_\partial -E_L, \varphi)| < C$ for any representation φ. For a knot K, we define $\rho^{(2)}(K) = \int \sigma_K(\omega)\, d\omega$, the integral of the Levine–Tristram signature function $\sigma_K(\omega)$ over the unit circle normalized to length 1. For example, an elementary calculation shows that if K is the trefoil, then $\rho^{(2)}(K) = \frac{4}{3}$ (see, for example, [**9**]). We choose knots J_i inductively for $i = 1, 2, \dots$ in such a way that the inequality

$$|\rho^{(2)}(J_i)| > C + N|\rho^{(2)}(J_j)| \tag{5}$$

is satisfied whenever $i > j$; recall that N is the number of satellite curves α_k. Once again, the connected sum of a sufficiently large number of trefoils can be taken as J_i. Let $J_{ik} = J_i$ for every k.

As we observed in the proof of Theorem 3.1, there is an integral homology equivalence $f\colon (E_{L_i}, \partial E_{L_i}) \to (E_L, \partial E_L)$. Since each α_k lies in $[\pi_1(E_L), \pi_1(E_L)]$, a Mayer–Vietoris argument applied to our satellite construction shows that f induces isomorphisms on $H_*(-;\mathbb{Z}[\mathbb{Z}^m])$.

Define $\alpha := \bigcup_{i=1}^{N} \alpha_i$. Let $\alpha_{ik} \subset E_{L\cup\alpha} \subset E_{L_i}$ be a push-off of α_k along the zero-framing. By the above, the $[\alpha_{ik}]$ generate $H_1(E_{L_i};\Lambda) \cong H_1(E_L;\Lambda)$. As we discussed in Section 2, $B\ell_{L_i} \cong B\ell_L$ under the map induced by f since f is a $\mathbb{Q}[\mathbb{Z}^m]$-homology isomorphism.

Suppose W is a 2-solvable cobordism between E_{L_i} and E_{L_j} for some $i \geqslant j$. We will show that this implies that $i = j$. We need the following fact, which is obtained immediately by combining [**3**, Theorem 4.12] with [**3**, Corollary 4.14], the proofs of which relied on arguments due to [**8**, Lemma 5.10; **9**, Theorem 4.4 and Lemma 4.5].

THEOREM 4.3 [**3**]. *Suppose that W is a 1-solvable cobordism between link exteriors E_L and $E_{L'}$. Then the submodule $P = \mathrm{Ker}\{tH_1(E_L;\Lambda) \to tH_1(W;\Lambda) \subset H_1(W;\Lambda)\}$ satisfies $B\ell_L(P,P) = 0$.*

In our case, from Theorem 4.3 and the hypothesis that $B\ell_L \cong B\ell_{L_i}$ is not constantly zero, it follows that $P = \mathrm{Ker}\{tH_1(E_{L_i};\Lambda) \to tH_1(W;\Lambda)\}$ is not equal to $tH_1(E_{L_i};\Lambda)$. That is, $[\alpha_{ik}] \notin P$ for some k.

For a group G, we now denote by $\{\mathcal{P}^n G\}$ the rational derived series, namely $\mathcal{P}^0 G := G$ and

$$\mathcal{P}^{n+1}G := \mathrm{Ker}\{\mathcal{P}^n G \to \frac{\mathcal{P}^n G}{[\mathcal{P}_n G, \mathcal{P}_n G]} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\}.$$

Then for $\pi = \pi_1(W)$, $\mathcal{P}^1\pi$ is the ordinary commutator subgroup $\pi^{(1)} = [\pi,\pi]$ since the abelian group $\pi/[\pi,\pi]$ is torsion free. Also, $\mathcal{P}^1\pi/\mathcal{P}^2\pi$ is the quotient of $\pi^{(1)}/\pi^{(2)} = H_1(W;\mathbb{Z}[\mathbb{Z}^m])$ by its $\mathbb{Z}$-torsion subgroup.

Let $\phi\colon \pi \to \Gamma := \pi/\mathcal{P}^2\pi$ be the projection, and by abuse of notation, we denote its restriction to $\partial W = E_{L_i} \cup_\partial -E_{L_j}$ by ϕ as well. Since $\mathcal{P}^1\pi/\mathcal{P}^2\pi = H_1(W;\mathbb{Z}[\mathbb{Z}^m])/(\mathbb{Z}\text{-torsion})$ injects into $H_1(W;\Lambda)$ and the image of $[\alpha_{ik}]$ in $H_1(W;\Lambda)$ is nontrivial for some k, it follows that $\phi([\alpha_{ik}])$ is nontrivial for some k. Furthermore, $\phi([\alpha_{ik}])$ has infinite order, since $\phi([\alpha_{ik}])$ lies in $\mathcal{P}^1\pi/\mathcal{P}^2\pi$, which is a torsion-free abelian group. Similarly $\phi([\alpha_{j\ell}])$ has infinite order for some ℓ.

Theorem 3.4 applies to the group Γ, with $\Gamma' = \mathcal{P}^1\pi/\mathcal{P}^2\pi$. We thus deduce that $\rho^{(2)}(E_{L_i} \cup_\partial -E_{L_j}, \phi) = 0$.

The homomorphism ϕ on $\pi_1(E_{L_i} \cup_\partial -E_{L_j})$ gives rise to a homomorphism $\varphi\colon \pi_1(E_L \cup_\partial -E_L) \to \Gamma$, which is defined in a very similar way to the map that was also called φ in the proof of Theorem 3.1: the homomorphism ϕ restricted to $\pi_1(E_{J_{ik}})$ sends the meridian of J_{ik} to $\phi([\alpha_{ik}])$. Since $\phi([\alpha_{ik}])$ has infinite order in the abelian subgroup $\mathcal{P}^1\pi/\mathcal{P}^2\pi$ of Γ, it follows that ϕ restricted to $E_{J_{ik}}$ factors as $\pi_1(E_{J_i}) \to \mathbb{Z} \hookrightarrow \Gamma$, where the first map is the abelianization. Similarly for $J_{j\ell}$. It follows that ϕ on $\pi_1(E_{L_i} \cup_\partial -E_{L_j})$ gives rise to a homomorphism of $\pi_1(E_L \cup_\partial -E_L)$, say φ.

By the above and by the same argument as in the proof of Theorem 3.1, we obtain that

$$\begin{aligned} 0 &= \rho^{(2)}(E_{L_i} \cup_\partial -E_{L_j}, \phi) \\ &= \rho^{(2)}(E_L \cup_\partial -E_L, \varphi) + \sum_{r=1}^{N} \epsilon_{ir}\rho^{(2)}(J_{ir},\psi) - \sum_{s=1}^{N} \epsilon_{js}\rho^{(2)}(J_{js},\psi), \end{aligned} \tag{6}$$

where ψ denotes the abelianization epimorphism of a knot group onto $\mathbb{Z}$ and where ϵ_{ir} is 0 if $\phi(\alpha_{ir})$ is trivial, 1 otherwise, and ϵ_{js} is defined similarly. Furthermore, by [**10**, Proposition 5.1] we know that for any knot K, the invariant $\rho^{(2)}(K,\psi)$ is equal to the integral of the Levine–Tristram signature function. Since $\epsilon_{ik} = 1 = \epsilon_{j\ell})$ and our J_i are chosen so that the inequality (5) is satisfied whenever $i > j$, the equality (6) can be satisfied only if $i = j$. From this, it follows that there is no 2-solvable cobordism between E_{L_i} and E_{L_j} whenever $i \neq j$. □

5. *Satellite construction and asymmetric Whitney towers*

In this section, we observe that our links L_i in Theorem B can be assumed to be mutually order n Whitney tower/grope concordant for any n. For the definition of order n Whitney tower concordance of framed links, see [**13**, Definition 3.2]; in this section, we assume that links are always endowed with the zero-framing.

Since we constructed L_i by satellite construction on a given link L using some knots which are only required to have sufficiently large integral (or average of finitely many evaluations) of the Levine–Tristram signature, the claim follows immediately from the following lemma:

Lemma 5.1. *Suppose K is a knot in S^3 with vanishing* Arf *invariant, L is a link in S^3, and α is a simple closed curve in $S^3 - L$ which is unknotted in S^3. Then the link $L' = L(\alpha, K)$ obtained by the satellite construction is order n Whitney tower/grope concordant to L for any n.*

For example, in the construction of our examples above, take an even number of trefoils for J_i. Note that asymmetric Whitney tower/grope concordance contains no information even when we use representations to nilpotent groups to obstruct symmetric Whitney tower concordance.

We remark that in [**32**], Schneiderman showed that the Whitney tower and grope concordance are equivalent in the asymmetric case. Therefore, it suffices to show our results for grope concordance. A brief outline of the proof is: K bounds an order n grope in D^4 since $\mathrm{Arf}(K) = 0$, and then a 'boundary connected sum' of parallel copies of this grope and the product concordance from L to L becomes an order n grope concordance between L and L'. The details are spelled out below.

Proof of Lemma 5.1. We begin with a well-known description of the satellite construction. Choose an embedded 2-disk D in S^3, which is bounded by α and meets L transversely. Choose an open regular neighborhood U of D in S^3 for which $(U, U \cap L)$ is a trivial r-string link where $r = |D \cap L|$. For the knot K, take the union Y of r parallel copies of K and take an open regular neighborhood V of a 2-disk fiber of the normal bundle of K such that $(V, V \cap Y)$ is a trivial r-string link. There is an orientation-reversing homeomorphism $h\colon (U, U \cap L) \to (V, V \cap Y)$ such that

$$(S^3, L') = (((S^3, L) - (U, U \cap L)) \cup ((S^3, Y) - (V, V \cap Y)))/x \sim h(x) \quad \text{for } x \in \partial U.$$

Here components of $Y - V$ are oriented according to the sign of the intersection points in $D \cap L$.

From our assumption that $\mathrm{Arf}(K) = 0$ and the result in [**31**] that the Arf invariant is the only obstruction to a knot bounding a framed embedded grope of arbitrary order, it follows that there is a framed embedded grope of order n in D^4 bounded by K for any n. Taking r parallel copies of the grope (and orienting the base surfaces according to the sign of the intersection points in $D \cap L$), we obtain a framed embedded grope G bounded by Y.

Identify a collar neighborhood of S^3 in D^4 with $S^3 \times [0, \epsilon]$. We may assume that $V \times [0, \epsilon]$ intersects (the base surface of) G in $(V \cap Y) \times [0, \epsilon]$. Now, define $G' \subset S^3 \times [0, 1]$ by forming the union

$$\begin{aligned}(S^3 \times [0,1], G') = (((S^3, L) \times [0,1] - (U, U \cap L) \times [0, \epsilon)) \\ \cup ((D^4, G) - (V, V \cap Y) \times [0, \epsilon)))/\sim,\end{aligned}$$

where $(x, t) \sim h(x, t)$ for $(x, t) \in (\partial U \times [0, \epsilon]) \cup (U \times \{\epsilon\})$. Then G' is a grope concordance of order n cobounded by $L' \times 0$ and $L \times 1$. □

Acknowledgements. The authors thank Kent Orr for helpful discussions. We are also grateful to the referee for carefully reading this paper.

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