A NOTE ON PARISIAN RUIN WITH AN ULTIMATE BANKRUPTCY LEVEL FOR LÉVY INSURANCE RISK PROCESSES

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Abstract. In this short paper, we investigate a definition of Parisian ruin introduced in [3], namely Parisian ruin with an ultimate bankruptcy level. We improve the results originally obtained and, moreover, we compute more general Parisian fluctuation identities.

1. Introduction

In classical ruin theory, the company is ruined when the surplus process falls below a critical threshold level. Inspired by Parisian options (see e.g. Chesney et al. [2]), some insurance risk models now consider the application of an implementation delay in the recognition of an insurer's capital insufficiency. More precisely, it is assumed that Parisian ruin occurs if the excursion below the critical threshold level is *too long*. The idea stems from the observation that in some industries, companies can continue to do business even though their wealth process falls below the critical level; see [11] for more motivation.

The idea of Parisian ruin has generated two types of models: with a deterministic implementation delay or a stochastic delay. The model with a deterministic delay has been studied in the Lévy setup by Czarna and Palmowski [4], Loeffen et al. [12] and more recently by Czarna [3], while Landriault et al. [10, 11] and Baurdoux et al. [1] have considered the idea of Parisian ruin with a stochastic implementation delay, with an emphasis on exponentially distributed delays.

In this paper, we study a general Lévy insurance risk model subject to Parisian ruin with an ultimate bankruptcy barrier, as defined in [3]. After calculating the probability of this type of Parisian ruin, we will derive a Parisian extention of the two-sided exit problem.

1.1. **Lévy insurance risk processes.**

Let $X = \{X_t, t \geq 0\}$ be a spectrally negative Lévy process (SNLP) on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$, that is a process with stationary and independent increments and no positive jumps. We exclude the case that X is the negative of a subordinator, i.e. we exclude the case of X having decreasing paths. In the actuarial ruin theory literature, processes as X are known as Lévy insurance risk processes. Note that the Cramér-Lundberg risk process and the Brownian approximation risk process belong to this family of stochastic processes. For more on the use of SNLPs in actuarial ruin theory, see e.g. [6, 8, 9].

The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $X_0 = 0$. The Laplace transform of X is given by

$$\mathbb{E}\left[\mathrm{e}^{\theta X_t}\right] = \mathrm{e}^{t\psi(\theta)},$$

for $\theta \geq 0$, where

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty \left(\mathrm{e}^{-\theta z} - 1 + \theta z \mathbf{1}_{(0,1]}(z)\right) \Pi(\mathrm{d}z),$$

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for $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and where Π is a measure on $(0,\infty)$ called the Lévy measure of X and such that

$$\int_0^\infty (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

Even though X has only negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line. Further, note that the *net profit condition* can be expressed as $\mathbb{E}[X_1] = \psi'(0+) > 0$. Note also that the process X has paths of bounded variation if and only if $\sigma = 0$ and $\int_0^1 z\Pi(\mathrm{d}z) < \infty$; this is the case when X is a Cramér-Lundberg process since then $\Pi(\mathrm{d}z) = \lambda F(\mathrm{d}z)$ where λ is the jump/claim rate of the underlying Poisson process and $F(\mathrm{d}z)$ is the jump/claim distribution.

1.2. **The idea of Parisian ruin.** Parisian ruin occurs if the excursion below the critical threshold level 0 is longer than a deterministic time called the implementation delay or the clock. It is worth pointing out that this definition of ruin is referred to as Parisian ruin due to its ties with Parisian options; see [2]. In [4, 12], a Parisian ruin time (with delay $r > 0$) is defined as

$$\kappa_r = \inf\{t > 0 \colon t - g_t > r\},$$

where $g_t = \sup\{0 \leq s \leq t \colon X_s \geq 0\}$. In other words, the company is said to be *Parisian ruined* the first time an excursion below zero lasts longer than the fixed implementation delay r. Therefore, $\mathbb{P}_x(\kappa_r < \infty)$ is the probability of Parisian ruin, when the initial capital is x, for which a nice and compact expression was obtained in [12]: if $\mathbb{E}[X_1] > 0$, then

$$\mathbb{P}_x(\kappa_r < \infty) = \begin{cases} 1 - \mathbb{E}[X_1]\frac{\int_0^\infty zW(x+z)\mathbb{P}(X_r \in \mathrm{d}z)}{\int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z)} & \text{for } x \geq 0, \\ 1 - \mathbb{E}[X_1]\frac{\mathbb{P}_x(\tau_0^+ \leq r)}{\int_0^\infty z\mathbb{P}(X_r \in \mathrm{d}z)} & \text{for } x < 0, \end{cases}$$

where W is the so-called 0-scale function of X (see the definition below) and τ_0^+ is the first passage time above 0.

Later in [3], a Parisian ruin time with a lower ultimate bankruptcy level was proposed. In this case, if the excursion below 0 is *too deep*, namely if the surplus goes below level $-a$, then even if the clock has not rung ruin is declared. For this more general stopping time, we first fix $a > 0$ and then define the Parisian ruin time with ultimate bankruptcy level $-a$ as

$$\kappa_r^a := \kappa_r \wedge \tau_{-a}^- = \min(\kappa_r, \tau_{-a}^-),$$

where τ_{-a}^- is the first passage time below $-a$. For this definition of Parisian ruin, a probabilistic decomposition was obtained in [3] for $\mathbb{P}_x(\kappa_r^a < \infty)$ and expressed in terms of the scale functions and the Lévy measure of X.

1.3. **Scale functions and fluctuation identities.** For an arbitrary SNLP with Laplace exponent ψ, there exists a function $\Phi \colon [0,\infty) \to [0,\infty)$ defined by $\Phi(q) = \sup\{\theta \geq 0 \mid \psi(\theta) = q\}$ such that $\psi(\Phi(q)) = q$. Note that we have $\Phi(q) = 0$ if and only if $q = 0$ and $\mathbb{E}[X_1] = \psi'(0+) \geq 0$.

We now recall the definition of the q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of the process X is such that $W^{(q)}(x) = 0$ for all $x < 0$ and is the unique continuous function on $[0,\infty)$ with Laplace transform given by

$$\int_0^\infty \mathrm{e}^{-\theta y} W^{(q)}(y)\mathrm{d}y = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q), \tag{1}$$

with the following definition for the initial value: $W^{(q)}(0) := \lim_{x \downarrow 0} W^{(q)}(x)$. This function is positive and strictly increasing on $[0,\infty)$. We write $W = W^{(0)}$ when $q = 0$. We will also frequently use the following functions: for $q \geq 0$ and $x \in \mathbb{R}$,

$$Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x), \tag{2}$$

where $\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(y)\mathrm{d}y$. It is known that

$$\lim_{x\uparrow\infty} \frac{W^{(q)}(x+y)}{W^{(q)}(x)} = \mathrm{e}^{\Phi(q)y}. \tag{3}$$

It was shown in [13] that, for $p, q \geq 0$ and $x \in \mathbb{R}$, we have

$$(q-p)\int_0^x W^{(p)}(x-y)W^{(q)}(y)\mathrm{d}y = W^{(q)}(x) - W^{(p)}(x) \tag{4}$$

and

$$(q-p)\int_0^x W^{(p)}(x-y)Z^{(q)}(y)\mathrm{d}y = Z^{(q)}(x) - Z^{(p)}(x). \tag{5}$$

We now present two *second-generation* scale functions which were introduced in [13]. First, for $p, p+q \geq 0$ and $x \in \mathbb{R}$, define

$$\begin{aligned} \mathcal{W}_a^{(p,q)}(x) &:= W^{(p+q)}(x) - q\int_0^a W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y \\ &= W^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y. \end{aligned} \tag{6}$$

Secondly, for $p \geq 0$, $q \in \mathbb{R}$ with $p+q \geq 0$ and $x \in \mathbb{R}$, define

$$\begin{aligned} \mathcal{Z}_a^{(p,q)}(x) &:= Z^{(p+q)}(x) - q\int_0^a Z^{(p+q)}(x-y)W^{(p)}(y)\mathrm{d}y \\ &= Z^{(p)}(x) + q\int_a^x W^{(p+q)}(x-y)Z^{(p)}(y)\mathrm{d}y. \end{aligned} \tag{7}$$

Note that $\mathcal{W}_a^{(p,q)}(a) = W^{(p)}(a)$ and $\mathcal{Z}_a^{(p,q)}(a) = Z^{(p)}(a)$.

Here is a collection of known fluctuation identities which will be used throughout this paper. For $c \in \mathbb{R}$, denote the following first passage times by

$$\tau_c^+ = \inf\{t > 0 : X_t > c\} \qquad \text{and} \qquad \tau_c^- = \inf\{t > 0 : X_t < c\}.$$

It is well known that, for $x \leq a$ and $q \geq 0$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbf{1}_{\{\tau_0^- < \tau_a^+\}}\right] = Z^{(q)}(x) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(x),$$

More generally, let us recall the following identities taken from [13]. For $p, q \geq 0$ and $y \leq a \leq x \leq b$, we have

$$\begin{aligned} \mathbb{E}_x\left[\mathrm{e}^{-p\tau_a^-}h(X_{\tau_a^-} - y)\mathbf{1}_{\{\tau_a^- < \tau_b^+\}}\right] &= h(x-y) - (q-p)\int_a^x W^{(p)}(x-z)h(z-y)\mathrm{d}z \\ &- \frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\left(h(b-y) - (q-p)\int_a^b W^{(p)}(b-z)h(z-y)\mathrm{d}z\right), \end{aligned} \tag{8}$$

for $h = W^{(q)}, Z^{(q)}$. Note that we can take the limit when b goes to infinity in this last result, separately for both possible functions represented by h; see e.g. [5].

The reader is referred to [7, 9] for more details on SNLPs and their fluctuation identities.

2. Parisian ruin with an ultimate bankruptcy barrier

We now present our main results. First, we compute the probability of Parisian ruin with an ultimate bankruptcy barrier and then we derive generalizations of the classical fluctuation identities related to the two-sided exit problem, when a Parisian delay is added.

Our first result is a semi-explicit expression for $\mathbb{P}_x\left(\kappa_r^a < \infty\right)$. It is an improvement over Theorems 1 and 2 in [3] in two ways: first, the cases of BV and UBV are unified, and secondly, it is expressed solely in terms of the scale functions of X (the Lévy measure does not appear in our expression).

Theorem 1. *Assume $\mathbb{E}[X_1] > 0$. For $r, a > 0$ and $x \geq -a$, we have*

$$\mathbb{P}_x\left(\kappa_r^a < \infty\right) = 1 - \mathbb{E}\left[X_1\right]\left\{W(x) + \frac{f_a(x;r)}{g_a(r)}\right\}, \tag{9}$$

where the Laplace transforms (with respect to r) of $g_a(r)$ and $f_a(x;r)$ are given by

$$\int_0^\infty \mathrm{e}^{-\theta r} g_a(r)\mathrm{d}r = \frac{Z^{(\theta)}(a)}{\theta W^{(\theta)}(a)} \tag{10}$$

and

$$\int_0^\infty \mathrm{e}^{-\theta r} f_a(x;r)\mathrm{d}r = \frac{\mathcal{W}_a^{(\theta,-\theta)}(x+a) - W(x)Z^{(\theta)}(a)}{\theta W^{(\theta)}(a)}. \tag{11}$$

As announced, our second result contains generalizations of the classical fluctuation identities related to the two-sided exit problem, when a Parisian delay is added.

Theorem 2. *For $q \geq 0$ and $r, a, b > 0$, if $-a \leq x \leq b$ then*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+ < \kappa_r^a\}}\right] = \frac{W^{(q)}(x) + f_a^{(q)}(x;r)/g_a^{(q)}(r)}{W^{(q)}(b)} \tag{12}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a < \tau_b^+\}}\right] = h_{a,b}^{(q)}(x;r) + \frac{n_{a,b}^{(q)}(r)}{g_{a,b}^{(q)}(r)} f_{a,b}^{(q)}(x;r), \tag{13}$$

where $f_a^{(q)}(x;r) = \lim_{b\to\infty} f_{a,b}^{(q)}(x;r)$, $g_a^{(q)}(r) = \lim_{b\to\infty} g_{a,b}^{(q)}(r)$, and where the Laplace transforms (with respect to r) of $f_{a,b}^{(q)}$, $n_{a,b}^{(q)}$, $g_{a,b}^{(q)}$ and $h_{a,b}^{(q)}$ are given by

$$\int_0^\infty \mathrm{e}^{-\theta r} f_{a,b}^{(q)}(x;r)\mathrm{d}r = \frac{1}{\theta}\left(\frac{\mathcal{W}_a^{(\theta+q,-\theta)}(x+a)}{W^{(\theta+q)}(a)} - \frac{W^{(q)}(x)}{W^{(q)}(b)}\frac{\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)}{W^{(\theta+q)}(a)}\right), \tag{14}$$

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r} n_{a,b}^{(q)}(r)\mathrm{d}r = &\left(\frac{Z^{(\theta+q)}(a)}{\theta W^{(\theta+q)}(a)} - \frac{\overline{W}^{(\theta+q)}(a)}{W^{(\theta+q)}(a)}\right)\frac{\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)}{W^{(q)}(b)} \\ &- \left(\frac{q}{\theta(\theta+q)}\right)\frac{\mathcal{Z}_a^{(\theta+q,-\theta)}(b+a)}{W^{(q)}(b)} - \frac{Z^{(q)}(a)}{(\theta+q)W^{(q)}(a)}\end{aligned} \tag{15}$$

$$\int_0^\infty \mathrm{e}^{-\theta r} g_{a,b}^{(q)}(r)\mathrm{d}r = \frac{\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)}{\theta W^{(\theta+q)}(a)W^{(q)}(b)} \tag{16}$$

and

$$\int_0^\infty \mathrm{e}^{-\theta r} h_{a,b}^{(q)}(x;r)\mathrm{d}r = \left(\frac{q}{\theta(\theta+q)}\right)\left\{\mathcal{Z}_a^{(\theta+q,-\theta)}(x+a) - \frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{Z}_a^{(\theta+q,-\theta)}(b+a)\right\} \tag{17}$$

$$-\left(\frac{Z^{(\theta+q)}(a)}{\theta W^{(\theta+q)}(a)}-\frac{\overline{W}^{(\theta+q)}(a)}{W^{(\theta+q)}(a)}\right)\left\{\mathcal{W}_a^{(\theta+q,-\theta)}(x+a)-\frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)\right\}$$
$$+\frac{1}{\theta+q}\left\{Z^{(q)}(x)-\frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(x)\right\}.$$

One can easily verify that

$$\lim_{b\to\infty}\frac{\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)}{W^{(q)}(b)}=\mathrm{e}^{\Phi(q)a}\left\{1+\theta\int_0^a \mathrm{e}^{-\Phi(q)y}W^{(\theta+q)}(y)\mathrm{d}y\right\},$$

where the expression on the right-hand-side corresponds to the function $\mathcal{H}^{(q,\theta)}(a)$ used in [1, 5, 13].

3. Proof of Theorem 1

Performing a standard probabilistic decomposition of the sample paths of X (see the fluctuation identities in Section 1.3), thanks to the strong Markov property and spectral negativity, we can write, for $-a\leq x<0$,

$$\mathbb{P}_x\left(\kappa_r\wedge\tau_{-a}^-=\infty\right)=\mathbb{P}_x\left(\tau_0^+<r\wedge\tau_{-a}^-\right)\mathbb{P}\left(\kappa_r\wedge\tau_{-a}^-=\infty\right),$$

where

$$\int_0^\infty \mathrm{e}^{-\theta r}\mathbb{P}_x\left(\tau_0^+<r\wedge\tau_{-a}^-\right)\mathrm{d}r=\frac{1}{\theta}\mathbb{E}_{x+a}\left[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right]=\frac{1}{\theta}\frac{W^{(\theta)}(x+a)}{W^{(\theta)}(a)}.$$

Consequently, for $x\geq 0$,

$$\begin{aligned}\mathbb{P}_x\left(\kappa_r\wedge\tau_{-a}^-=\infty\right)&=\mathbb{P}_x\left(\tau_0^-=\infty\right)+\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left(\kappa_r\wedge\tau_{-a}^-=\infty\right)\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\\&=\mathbb{E}\left[X_1\right]W(x)+\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left(\tau_0^+<r\wedge\tau_{-a}^-\right)\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\mathbb{P}\left(\kappa_r\wedge\tau_{-a}^-=\infty\right).\end{aligned}\tag{18}$$

In fact, it is easy to verify that this last expression is valid for any $x\geq -a$.

If we assume that X is of BV, then, setting $x=0$ in (18), we get

$$\mathbb{P}\left(\kappa_r^a=\infty\right)=\mathbb{P}\left(\kappa_r\wedge\tau_{-a}^-=\infty\right)=\mathbb{E}\left[X_1\right]\frac{W(0)}{1-\mathbb{E}\left[\mathbb{P}_{X_{\tau_0^-}}\left(\tau_0^+<r\wedge\tau_{-a}^-\right)\mathbf{1}_{\{\tau_0^-<\infty\}}\right]},$$

where the Laplace transform (in r) of the expectation at the denominator can be computed. To this end, we set

$$f_a(x;r):=\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left(\tau_0^+<r\wedge\tau_{-a}^-\right)\mathbf{1}_{\{\tau_0^-<\infty\}}\right].$$

Hence, using Fubini's theorem and the above computations, for any $x\geq -a$,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r}f_a(x;r)\mathrm{d}r&=\frac{1}{\theta W^{(\theta)}(a)}\mathbb{E}_x\left[W^{(\theta)}(X_{\tau_0^-}+a)\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\\&=\frac{1}{\theta W^{(\theta)}(a)}\left\{W(x+a)+\theta\int_0^a W(x+a-y)W^{(\theta)}(y)\mathrm{d}y-W(x)Z^{(\theta)}(a)\right\},\end{aligned}$$

where in the last equality we used Equation (8) with $b\to\infty$, equations (2), (3), (6) and the fact that $\Phi(0)=0$ under the assumption $\mathbb{E}\left[X_1\right]>0$. In particular, when $x=0$, using Equation (4), we have

$$\int_0^\infty \mathrm{e}^{-\theta r}g_a(r)\mathrm{d}r=\frac{Z^{(\theta)}(a)}{\theta W^{(\theta)}(a)},$$

where $g_a(r) := (1 - f_a(0;r))/W(0)$, as announced in Equation (10). In conclusion, if X is of BV and $x = 0$, the result of Equation (9) follows.

For an arbitrary value of $x \geq -a$, plugging the above in (18), one obtains

$$\mathbb{P}_x\left(\kappa_r^a < \infty\right) = \mathbb{P}_x\left(\kappa_r \wedge \tau_{-a}^- < \infty\right) = 1 - \mathbb{E}\left[X_1\right]\left\{W(x) + \frac{f_a(x;r)}{g_a(r)}\right\},$$

where the Laplace transforms (with respect to r) of $f_a(x;r)$ and $g_a(r)$ are given in (11) and (10), respectively.

If X is a general SNLP (not necessarily of BV), as in [1, 12], we can use the following limiting argument. First, for $\epsilon \geq 0$, we define $\kappa_{r,\epsilon}$ as the first time that an excursion below zero, which has reached level $-\epsilon$, lasts longer than the fixed implementation delay r. Clearly, we have $\kappa_{r,0} = \kappa_r$. As in [1], one can prove that

$$\kappa_{r,\epsilon} \underset{\varepsilon \downarrow 0}{\longrightarrow} \kappa_r, \quad \mathbb{P}\text{-a.s.}$$

For the rest of the proof, we assume $\epsilon > 0$. Similarly as above, we can show that

$$\mathbb{P}\left(\kappa_{r,\epsilon}^a = \infty\right) = \mathbb{P}\left(\kappa_{r,\epsilon} \wedge \tau_{-a}^- = \infty\right) = \mathbb{E}\left[X_1\right] \frac{W(\epsilon)}{1 - \mathbb{E}\left[\mathbb{P}_{X_{\tau_{-\epsilon}^-}}\left(\tau_0^+ < r \wedge \tau_{-a}^-\right) \mathbf{1}_{\{\tau_{-\epsilon}^- < \infty\}}\right]},$$

where by l'Hôpital's rule

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta r}\left(1/W(\epsilon)\right)\left(1 - \mathbb{E}\left[\mathbb{P}_{X_{\tau_{-\epsilon}^-}}\left(\tau_0^+ < r \wedge \tau_{-a}^-\right) \mathbf{1}_{\{\tau_{-\epsilon}^- < \infty\}}\right]\right) \mathrm{d}r \\ = \frac{\int_{a-\epsilon}^a W(a-y) W^{(\theta)}(y) \mathrm{d}y}{W(\epsilon) W^{(\theta)}(a)} + \frac{Z^{(\theta)}(a-\epsilon)}{\theta W^{(\theta)}(a)} \underset{\varepsilon \downarrow 0}{\longrightarrow} \frac{Z^{(\theta)}(a)}{\theta W^{(\theta)}(a)}.\end{aligned}$$

By continuity of Laplace transforms, we can use this result in Equation (18) (which is valid for any SNLP), and then the result follows.

Note that if X is of UBV and $x = 0$, using Equation (4) we can show that

$$\int_0^\infty \mathrm{e}^{-\theta r} f_a(0;r) \mathrm{d}r = \frac{1}{\theta},$$

which means that $f_a(0;r) = 1$, for all $r > 0$. Therefore, the result in (9) holds whether X is of BV or of UBV.

4. Proof of Theorem 2

Performing a standard probabilistic decomposition of the sample paths of X using the strong Markov property and spectral negativity, together with the fluctuation identities in Section 1.3, we can write, for $-a \leq x \leq b$,

(19)

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \kappa_r^a\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)} + \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} \mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ < r \wedge \tau_{-a}^-\}}\right] \mathbf{1}_{\{\tau_0^- < \infty\}}\right] \mathbb{E}\left[\mathrm{e}^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \kappa_r^a\}}\right].$$

Note that we used the fact that X is skip-free upward. Define $f_a^{(q)}(x;r)$ by

$$f_a^{(q)}(x;r) := \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} \mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ < r \wedge \tau_{-a}^-\}}\right] \mathbf{1}_{\{\tau_0^- < \infty\}}\right].$$

Note that $f_a^{(q)}(x;r) = \lim_{b \to \infty} f_{a,b}^{(q)}(x;r)$, where

$$f_{a,b}^{(q)}(x;r) := \mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-} \mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ < r \wedge \tau_{-a}^-\}}\right] \mathbf{1}_{\{\tau_0^- < \tau_b^+\}}\right].$$

As in the proof of Theorem 1, if we assume that X is of BV, then, setting $x=0$ yields

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\kappa_r^a\}}\right]=\frac{1/g_a^{(q)}(r)}{W^{(q)}(b)},$$

where $g_a^{(q)}(r):=(1-f_a^{(q)}(0;r))/W^{(q)}(0)$ has Laplace transform given in Equation (16) when $b\to\infty$. In conclusion, when X is of BV, the result of (12) is verified. If X is a general SNLP (not necessarily of BV), we can use the the same limiting argument as in Theorem 1. The details are left to the reader.

Performing again a standard probabilistic decomposition of the sample paths of X using the strong Markov property and spectral negativity, together with the fluctuation identities in Section 1.3, we can write, for $0\le x\le b$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right]=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right],$$

where, for $-a\le x<0$,

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right]&=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_{-a}^-}\mathbf{1}_{\{\tau_{-a}^-<\tau_0^+\wedge r\}}\right]+\mathrm{e}^{-qr}\mathbb{P}_x\left(r<\tau_{-a}^-\wedge\tau_0^+\right)\\&\quad+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\wedge\tau_{-a}^-\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right].\end{aligned}$$

Putting the pieces together, we obtain, for all $-a\le x\le b$,

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right]&=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_{-a}^-}\mathbf{1}_{\{\tau_{-a}^-<\tau_0^+\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\\&\quad+\mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{P}_{X_{\tau_0^-}}\left(r<\tau_{-a}^-\wedge\tau_0^+\right)\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\\&\quad+\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_0^+}\mathbf{1}_{\{\tau_0^+<r\wedge\tau_{-a}^-\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right].\end{aligned}\tag{20}$$

Define $h_{a,b}^{(q)}(x;r)$ by

$$\begin{aligned}h_{a,b}^{(q)}(x;r):=\mathbb{E}_x&\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\mathrm{e}^{-q\tau_{-a}^-}\mathbf{1}_{\{\tau_{-a}^-<\tau_0^+\wedge r\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\\&+\mathrm{e}^{-qr}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{P}_{X_{\tau_0^-}}\left(r<\tau_{-a}^-\wedge\tau_0^+\right)\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right].\end{aligned}$$

We can now re-write Equation (20) as

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right]=h_{a,b}^{(q)}(x;r)+f_{a,b}^{(q)}(x;r)\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right].\tag{21}$$

We will analyze each part of the above decomposition separately. Again, using Fubini's theorem, the fluctuation identities of Section 1.3 and identities (8), (4) and (5), we can compute the following Laplace transforms. First, we have

$$\begin{aligned}\int_0^\infty\mathrm{e}^{-\theta r}h_{a,b}^{(q)}(x;r)\mathrm{d}r=\left(\frac{q}{\theta(\theta+q)}\right)&\left\{\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}Z^{(\theta+q)}(X_{\tau_0^-}+a);\tau_0^-<\tau_b^+\right]\right.\\-\left(\frac{Z^{(\theta+q)}(a)}{\theta W^{(\theta+q)}(a)}-\frac{\overline{W}^{(\theta+q)}(a)}{W^{(\theta+q)}(a)}\right)&\left.\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}W^{(\theta+q)}(X_{\tau_0^-}+a);\tau_0^-<\tau_b^+\right]\right\}\\&+\frac{1}{\theta+q}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-};\tau_0^-<\tau_b^+\right]\end{aligned}$$

$$= \left(\frac{q}{\theta(\theta+q)}\right)\left\{\mathcal{Z}_a^{(\theta+q,-\theta)}(x+a) - \frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{Z}_a^{(\theta+q,-\theta)}(b+a)\right\}$$

$$- \left(\frac{Z^{(\theta+q)}(a)}{\theta W^{(\theta+q)}(a)} - \frac{\overline{W}^{(\theta+q)}(a)}{W^{(\theta+q)}(a)}\right)\left\{\mathcal{W}_a^{(\theta+q,-\theta)}(x+a) - \frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)\right\}$$

$$+ \frac{1}{\theta+q}\left\{Z^{(q)}(x) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(x)\right\}.$$

And also,

$$\int_0^\infty \mathrm{e}^{-\theta r} f_{a,b}^{(q)}(x;r)\mathrm{d}r = \frac{1}{\theta W^{(\theta+q)}(a)}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}W^{(\theta+q)}(X_{\tau_0^-}+a)\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]$$

$$= \frac{1}{\theta W^{(\theta+q)}(a)}\left(\mathcal{W}_a^{(\theta+q,-\theta)}(x+a) - \frac{W^{(q)}(x)}{W^{(q)}(b)}\mathcal{W}_a^{(\theta+q,-\theta)}(b+a)\right).$$

Note that if X has paths of UBV, then $f_{a,b}^{(q)}(0;r) = 1$.

Now, if we assume that X is of BV, then, setting $x = 0$ in (21), we get

$$\mathbb{E}\left[\mathrm{e}^{-q\kappa_r^a}\mathbf{1}_{\{\kappa_r^a<\tau_b^+\}}\right] = \frac{n_{a,b}^{(q)}(r)}{g_{a,b}^{(q)}(r)},$$

where $n_{a,b}^{(q)}(r) := h_{a,b}^{(q)}(0;r)/W^{(q)}(0)$ and $g_{a,b}^{(q)}(r) := (1 - f_{a,b}^{(q)}(0;r))/W^{(q)}(0)$ with Laplace transforms (with respect to r) given in Equations (15) and (16), respectively. In conclusion, putting all the pieces together, when X is of BV, the result of (13) is verified. If X is a general SNLP, we can use the same limiting argument as above. Again, the details are left to the reader.

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