A parallelogram tile fills the plane by translation in at most two distinct ways ☆

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Abstract

We consider the tilings by translation of a single polyomino or tile on the square grid $\mathbb{Z}^2$. It is well-known that there are two regular tilings of the plane, namely, parallelogram and hexagonal tilings. Although there exist tiles admitting an arbitrary number of distinct hexagon tilings, it has been conjectured that no polyomino admits more than two distinct parallogram tilings. In this paper, we prove this conjecture.

Keywords: Tilings, polyominoes

1. Introduction

Tilings appeared as one of the archetypes of the close relationship between art and mathematics, and are present in human history under various representations. The beautiful book of Grünbaum and Shephard [1] contains a systematic study of tilings, presenting a number of challenging problems (see also [2] for related work). For instance, the problem of designing an efficient algorithm for deciding whether a given polygon tiles the plane becomes

☆With support of NSERC (Canada).

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more tractable when restricted to polyominoes, that is, subsets of the square lattice $\mathbb{Z}^2$ whose boundary is a non-crossing closed path. Indeed, while a sufficient condition is provided by the *Conway criterion* in [3], the boundary of such tiles must be composed of matching pairs of opposite sides which interlock when translated and there might be either two or three such pairs (see [3] p. 225 for more details). Beauquier and Nivat [4] established that this condition was also necessary for tiling by translation in two directions, so that such objects are generalizations of parallelograms and parallel hexagons, hexagons whose opposite sides are parallel. In other words, these tiles are continuous deformations of either the *unit square* or the *regular hexagon*. Here, we consider tilings obtained by translation of a single polyomino, called *exact* in [4]. Paths are conveniently described by words on the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$, representing the elementary grid steps $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$. Beauquier and Nivat [4] characterized exact polyominoes by showing that the boundary word $b(P)$ of such a polyomino satisfies the equation $b(P) = X \cdot Y \cdot Z \cdot \widehat{X} \cdot \widehat{Y} \cdot \widehat{Z}$, where at most one of the variables is empty and where $\widehat{W}$ is the path W traveled in the opposite direction. From now on, this condition is referred as the *BN-factorization*. An exact polyomino is said to be a *hexagon* if none of the variables X, Y, Z is empty and a *square* if one of them is so. While decidability was already established in [5], recently, it was shown that a linear algorithm exists for deciding whether a word $w \in \mathcal{F}$ represents a square or not. It is based on data structures that include radix-trees, for checking that w is a closed non crossing path [6], and suffix-trees for extracting the BN-factorization [7].

Observe that a single polyomino may lead to many regular tilings (spanned by two translation vectors) of the plane. For instance the $n \times 1$ rectangle does it in $n-1$ distinct ways as a hexagon (see Figure 1). On the other hand,

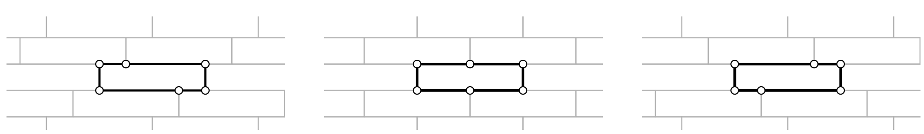


Figure 1: The three hexagonal tilings of the 4×1 rectangle.

square factorizations are more constrained and it was conjectured by Brlek, Dulucq, Fédou and Provençal (reported in [8]) that an exact polyomino tiles the plane as a square in at most two distinct ways. Squares having exactly two distinct BN-factorizations are called *double squares*. For instance, Christoffel and Fibonacci tiles introduced recently [9] are examples of double squares (Figure 2). See [10] for connections between Fibonacci tiles and number

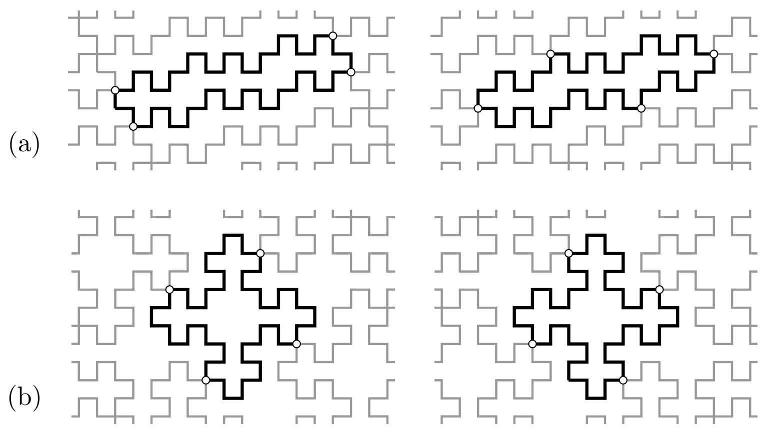


Figure 2: (a) A Christoffel tile yields two distinct non-symmetric square tilings of the plane. (b) The Fibonacci tile of order 2 and its two symmetric square tilings.

theory. Our main result is the proof of the double square conjecture [8].

Theorem 1. *Every polyomino yields at most two distinct square tilings.*

Note that there are words having more than two square BN-factorizations. An example of length 36 (in fact a shortest one, up to conjugacy) was provided by Provençal [11]:

3	**3**	**011**	**03301**	**10330**	**110**	**3**	**3**	**211**	**23321**	**12332**	**112**	**3**	**3**
U			V			$\widehat{U}$			$\widehat{V}$				
	X			Y			$\widehat{X}$			$\widehat{Y}$			
		W			Z			$\widehat{W}$			$\widehat{Z}$		

However, this word does not code the boundary of a polyomino as it is intersecting (see Figure 3). Hence, solving only equations on words is not sufficient for our purpose. Our approach uses geometrical and topological properties of the boundary word that are deduced from the equations.

2. Preliminaries

The usual terminology and notation on words is from Lothaire [12]. An *alphabet* $\mathcal{A}$ is a finite set whose elements are *letters*. A finite word w is a

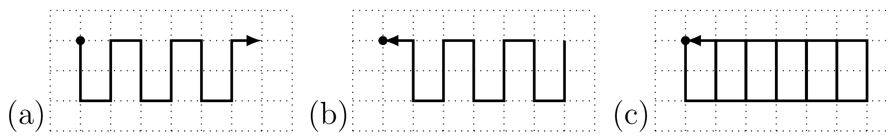


Figure 3: the paths (a) UV and (b) $\widehat{U}\widehat{V}$. The path (c) $UV\widehat{U}\widehat{V}$ has 3 distinct square factorizations but it intersects itself.

function $w : [1, 2, \ldots, n] \to \mathcal{A}$, where w_i is the i-th letter, $1 \leq i \leq n$. For later use, we define the auxiliary functions $\mathrm{First}(w) = w_1$ and $\mathrm{Last}(w) = w_n$. The *length* of w, denoted by $|w|$, is the integer n. The length of the empty word ε is 0. The *free monoid* $\mathcal{A}^*$ is the set of all finite words over $\mathcal{A}$. The *reversal* of $w = w_1w_2\cdots w_n$ is the word $\widetilde{w} = w_nw_{n-1}\cdots w_1$. A word u is a *factor* of another word w if there exist $x, y \in \mathcal{A}^*$ such that $w = xuy$. When $x, y \neq \varepsilon$, u is called *proper factor* of w. We denote by $|w|_u$ the number of occurrences of u in w. Two words u and v are *conjugate*, written $u \equiv v$ or sometimes $u \equiv_{|x|} v$, when x, y are such that $u = xy$ and $v = yx$. Conjugacy is an equivalence relation and the class of a word w is denoted $[w]$. In this paper, the alphabet $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$ is identified with $\mathbb{Z}_4$, the additive group of integers mod 4. This allows to use the basic transformations on $\mathbb{Z}_4$, namely, rotations $\rho^i : x \mapsto x + i$ and reflections $\sigma_i : x \mapsto i - x$, as maps on $\mathcal{F}$ which extend uniquely to morphisms (w.r.t concatenation) on $\mathcal{F}^*$. Given a nonempty word $w \in \mathcal{F}^*$, the *first differences word* $\Delta(w) \in \mathcal{F}^*$ of w is ε if $|w| = 1$, and otherwise

$$\Delta(w) = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}). \tag{1}$$

One may verify that if $w, z \in \mathcal{F}^* \setminus \{\varepsilon\}$, then $\Delta(wz) = \Delta(w)\Delta(w_nz_1)\Delta(z)$. Words in $\mathcal{F}^*$ are interpreted as paths in the square grid, so that we indistinctly talk of any word $w \in \mathcal{F}^*$ as the *path* w. Moreover, the word

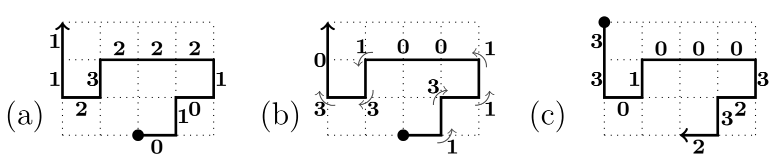


Figure 4: (a) The path $w = \mathbf{01012223211}$. (b) Its first differences word $\Delta(w) = \mathbf{1311001330}$. (c) Its homologous $\widehat{w} = \mathbf{33010003232}$.

$\widehat{w} := \rho^2(\widetilde{w})$ is *homologous* to w, i.e., in direction opposite to that of w

(Figure 4). A word $u \in \mathcal{F}^*$ may contain factors in $\mathcal{C} = \{\mathbf{02}, \mathbf{20}, \mathbf{13}, \mathbf{31}\}$, corresponding to cancelling steps on a path. Nevertheless, each word w can be reduced in a unique way to a word w', by sequentially applying the rewriting rules of the form $u \mapsto \varepsilon$, for $u \in \mathcal{C}$. The *reduced word* w' of w is nothing but a word in $\mathcal{P} = \mathcal{F}^* \setminus \mathcal{F}^*\mathcal{C}\mathcal{F}^*$. We define the *turning number*[1] of w by $\mathcal{T}(w) = (|\Delta(w')|_{\mathbf{1}} - |\Delta(w')|_{\mathbf{3}})/4$.

A path w is *closed* if it satisfies $|w|_{\mathbf{0}} = |w|_{\mathbf{2}}$ and $|w|_{\mathbf{1}} = |w|_{\mathbf{3}}$, and it is *simple* if no proper factor of w is closed. A *boundary word* is a simple and closed path, and a *polyomino* is a subset of $\mathbb{Z}^2$ contained in some boundary word. It is convenient to represent each closed path w by its conjugacy class $[w]$, also called *circular word*. An adjustment is necessary to the function $\mathcal{T}$, for we take into account the closing turn. The first differences also noted Δ is defined on any closed path w by setting

$$\Delta([w]) \equiv \Delta(w) \cdot (w_1 - w_n),$$

which is also a closed word. By applying the same rewriting rules, a circular word $[w]$ is *circularly-reduced* to a unique word $[w']$. If w is a closed path, then the *turning number*[1] of w is

$$\mathring{\mathcal{T}}(w) = \mathcal{T}([w]) = (|\Delta([w'])|_{\mathbf{1}} - |\Delta([w'])|_{\mathbf{3}})/4.$$

It corresponds to its total curvature divided by 2π. Clearly, the turning number $\mathcal{T}([w])$ of a closed path w belongs to $\mathbb{Z}$ (see [13, 14]), and in particular, the Daurat-Nivat relation [15] may be rephrased as follows.

Proposition 2. *The turning number of a boundary word w is $\mathring{\mathcal{T}}(w) = \pm 1$.*

Now, we may define orientation: a boundary word w is *positively oriented* (counterclockwise) if its turning number is $\mathring{\mathcal{T}}(w) = 1$. As a consequence, every square satisfies the following factorization

Lemma 3. *Let $w \equiv XY\widehat{X}\widehat{Y}$ be the boundary word of a square, then*

$$\Delta([w]) \equiv \Delta(X) \cdot \alpha \cdot \Delta(Y) \cdot \alpha \cdot \Delta(\widehat{X}) \cdot \alpha \cdot \Delta(\widehat{Y}) \cdot \alpha,$$

where $\alpha = \mathbf{1}$ if w is positively oriented, and $\alpha = \mathbf{3}$ otherwise.

[1] In [13, 14], the authors introduced the notion of *winding number* of w which is $4\mathcal{T}(w)$.

Proof. The equation $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ holds for all $w \in \mathcal{F}^*$ and the turning number of a positively oriented boundary word is 1. $\square$

The next property is easy to check.

Lemma 4. *Let $w \equiv XY\widehat{X}\widehat{Y}$ be a boundary word of a square. Then* $\text{First}(X) = \text{Last}(X)$ *and* $\text{First}(Y) = \text{Last}(Y)$.

Proof. By Proposition 3, we have $\text{First}(X) - \text{Last}(\widehat{Y}) = \text{First}(Y) - \text{Last}(X) = \text{First}(\widehat{X}) - \text{Last}(Y) \in \{\mathbf{1}, \mathbf{3}\}$. Since $\text{Last}(\widehat{Y}) = \rho^2(\text{First}(Y))$ and $\text{First}(\widehat{X}) = \rho^2(\text{Last}(X))$, we deduce that

$$\begin{aligned} \text{First}(X) - \rho^2(\text{First}(Y)) &= \text{First}(Y) - \text{Last}(X) \\ &= \rho^2(\text{Last}(X)) - \text{Last}(Y) \in \{\mathbf{1}, \mathbf{3}\}. \end{aligned}$$

By summing up those last equalities, since $\alpha - \rho^2(\alpha) = \mathbf{2}$ for all letters $\alpha \in \mathcal{F}$ and since $\mathbf{1} + \mathbf{1} = \mathbf{3} + \mathbf{3} = \mathbf{2}$, we obtain $\text{First}(X) - \text{Last}(X) + \mathbf{2} = \mathbf{2}$ and $\text{First}(Y) - \text{Last}(Y) + \mathbf{2} = \mathbf{2}$, and the result follows. $\square$

We end this section with a useful result adapted from [7, 8]. Indeed, the core of the proof of our main result is based on the fact that if a polyomino has two distinct square factorizations, then they alternate, i.e. no factor of one factorization is included in a factor of the other one (see Corollary 6 in [7]).

Lemma 5. [7, 8] *Let w be the boundary word of an exact polyomino P. If w satisfies*

$$w \equiv UV\widehat{U}\widehat{V} = \alpha XY\widehat{X}\beta$$

with $\widehat{Y} = \beta\alpha$ and $\beta \neq \varepsilon$, then either

(i) *$\alpha = \varepsilon$ and $U = X, V = Y$ and the factorizations coincide, or*

(ii) *$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$, with $0 < d_1 < |U| < d_1 + |X|$.*

3. Proof of Theorem 1

In this section, we assume that there exists a polyomino that tiles the plane as a square in three ways, i.e., its positively oriented boundary word has three distinct square factorizations given by

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}. \tag{2}$$

By Lemma 5, the factorizations must alternate which translates into the inequalities

$$0 < d_1 < d_1 + d_2 < |U| < d_1 + |X| < d_1 + d_2 + |W|,$$

and we have the situation depicted in Figure 5-a.

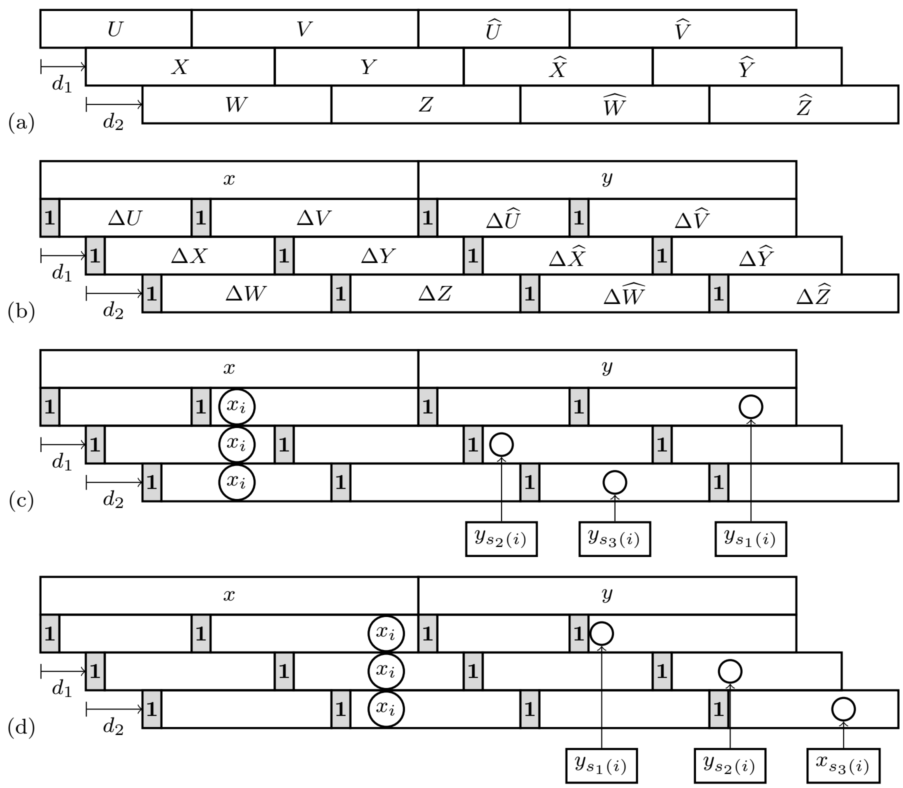


Figure 5: (a) Three distinct square factorizations of a tile. Note that $0 < d_1 < d_1 + d_2 < |U| < d_1 + |X| < d_1 + d_2 + |W|$. they alternate. (b) The corresponding first differences. (c) The images of the position i in x by the reflections s_1, s_2 and s_3. The letters at these positions are related by the relations $x_i = \overline{y_{s_1(i)}} = \overline{y_{s_2(i)}} = \overline{y_{s_3(i)}}$. (d) The images of the position i in x by the reflections s_2 or s_3 (but not s_1) can also be to the right of y thus inside x. In this case, we have the relation $x_i = \overline{y_{s_1(i)}} = \overline{y_{s_2(i)}} = \overline{x_{s_3(i)}}$.

Let $I = \{0, d_1, d_1 + d_2, |U|, d_1 + |X|, d_1 + d_2 + |W|\}$ be the set of six *corners* of the boundary. It follows from Lemma 5 that all these corners are distinct, that is $|I| = 6$. Furthermore, it is convenient to consider the first differences word of the boundary word as two parts

$$\begin{aligned} x &= x_0x_1x_2\cdots x_{n-1} &= \mathbf{1}\cdot\Delta U\cdot\mathbf{1}\cdot\Delta V, \\ y &= y_0y_1y_2\cdots y_{n-1} &= \mathbf{1}\cdot\Delta\widehat{U}\cdot\mathbf{1}\cdot\Delta\widehat{V}, \end{aligned}$$

where $n = |x| = |y|$ is the half-perimeter. Note that $x_i = y_i = \mathbf{1}$ for all six corners $i \in I$ (see Figure 5-b). Three reflections on $\mathbb{Z}_n$ are useful:

$$\begin{aligned} s_1 : i &\mapsto (|U| - i) \mod n, \\ s_2 : i &\mapsto (|X| + 2d_1 - i) \mod n, \\ s_3 : i &\mapsto (|W| + 2(d_1 + d_2) - i) \mod n. \end{aligned}$$

They satisfy $s_1^2 = s_2^2 = s_3^2 = 1$ and $(s_js_ks_\ell)^2 = 1$ for all $j, k, \ell \in \{1, 2, 3\}$ which is equivalent to the following identity:

$$s_ks_\ell s_js_ks_\ell = s_j. \tag{3}$$

If $(s_js_k)^2 = 1$ with $s_j \neq s_k$ then s_j and s_k are *perpendicular*. From Lemma 5, the reflections s_1, s_2 and s_3 are pairwise distinct. We say that s_1 is *admissible* on i if $i \notin \{0, |U|\}$ and similarly for s_2 if $i \notin \{d_1, |X| + d_1\}$ and for s_3 if $i \notin \{d_1 + d_2, |W| + d_1 + d_2\}$. Below we denote $\overline{\alpha} := \sigma_0(\alpha)$ so that $\overline{\mathbf{0}} = \mathbf{0}$, $\overline{\mathbf{1}} = \mathbf{3}$, $\overline{\mathbf{2}} = \mathbf{2}$ and $\overline{\mathbf{3}} = \mathbf{1}$. The fact that $(\Delta w)_i = \overline{(\Delta\widehat{w})_{|w|-i}}$ for all $w \in \{U, V, X, Y, W, Z\}$ and $1 \leq i \leq |w| - 1$ then translates nicely in terms of x, y and reflections s_1, s_2 and s_3 (see Figure 5-c-d).

Lemma 6. *Let $i \in \mathbb{Z}_n$ and $j \in \{1, 2, 3\}$ such that s_j is admissible on i. Then*

(i) $y_i = \overline{x_{s_j(i)}}$ *and* $x_i = \overline{y_{s_j(i)}}$ *or* $x_i = \overline{x_{s_j(i)}}$ *and* $y_i = \overline{y_{s_j(i)}}$.

(ii) *If $x_i = y_i$, then $x_{s_j(i)} = y_{s_j(i)}$.*

Proof. (i) There are three cases to consider according to the value of j. First, supppose $j = 1$ and assume that $0 < i < |U|$. We have

$$x_i = (\Delta U)_i = \overline{(\Delta\widehat{U})_{|U|-i}} = \overline{y_{s_j(i)}},$$

$$y_i = (\Delta\widehat{U})_i = \overline{(\Delta U)_{|U|-i}} = \overline{x_{s_j(i)}}.$$

On the other hand, if $|U| < i < n$, then $s_j(i) = |U| - i + n$ and

$$x_i = (\Delta V)_{i-|U|} = \overline{(\Delta\widehat{V})_{n-i}} = \overline{y_{s_j(i)}},$$

$$y_i = (\Delta\widehat{V})_{i-|U|} = \overline{(\Delta V)_{n-i}} = \overline{x_{s_j(i)}}.$$

Now, supppose $j = 2$ and assume that $0 < i < d_1$. We have

$$x_i = (\Delta\widehat{Y})_{|Y|+i-d_1} = \overline{(\Delta Y)_{d_1-i}} = \overline{x_{2d_1+|X|-i}} = \overline{x_{s_j(i)}},$$

$$y_i = (\Delta Y)_{|Y|+i-d_1} = \overline{(\Delta\widehat{Y})_{d_1-i}} = \overline{y_{2d_1+|X|-i}} = \overline{y_{s_j(i)}}.$$

The other cases for $j = 2$ and $j = 3$ are similar.
(ii) If $x_i = y_i$, then (i) implies $x_{s_j(i)} = y_{s_j(i)}$. □

Note that if s_j is not admissible on i and $k \neq j$, then s_k must be admissible on i, because I contains 6 distinct corners. We say that a sequence of reflections $(s_{j_m}, \ldots, s_{j_2}, s_{j_1})$ is *admissible* on i if each s_{j_k} is admissible on $s_{j_{k-1}} \cdots s_{j_2} s_{j_1}(i)$. By abuse of notation, we equivalently write that the expression $s_{j_m} \cdots s_{j_2} s_{j_1}$ is *admissible* on i.

Lemma 7. *Let $i \in I$ and $S = s_{j_m} s_{j_{m-1}} \cdots s_{j_2} s_{j_1}$ be an admissible product of reflections on i. Then $x_{S(i)} = y_{S(i)}$ and*

$$x_{S(i)} = \begin{cases} x_i & \text{if } m \text{ is even,} \\ \overline{x_i} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By induction on m and from Lemma 6. □

Lemma 8. *The following statements hold.*

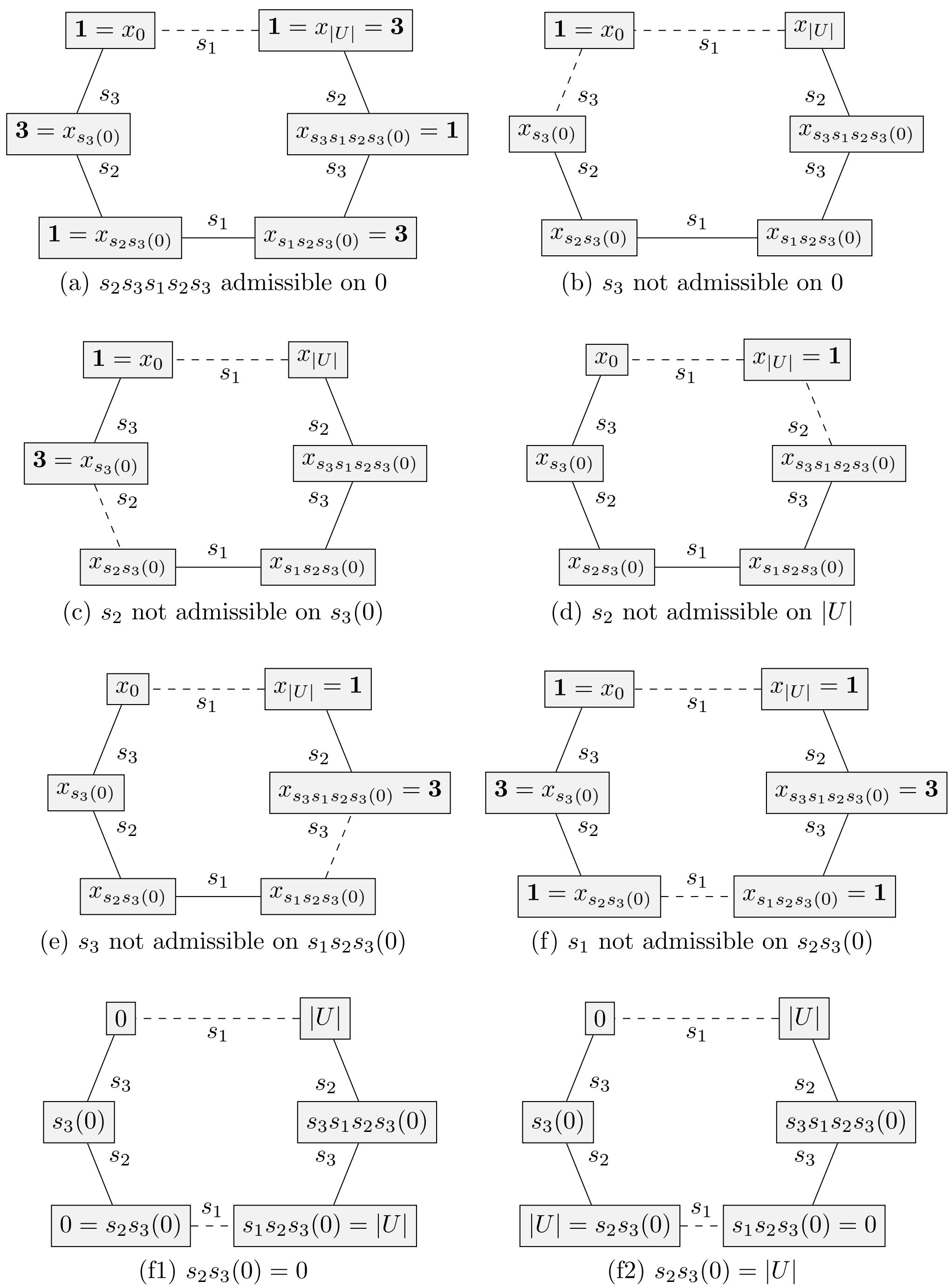
(i) $s_2 s_3(0) = |U| = s_3 s_2(0)$.

(ii) $s_1 s_3(d_1) = d_1 + |X| = s_3 s_1(d_1)$.

Proof. The proof proceeds by examining several cases. In each case we reach a contradiction by showing that either two distinct reflections are not admissible on the same position, or that the letter **3** occurs on a corner, or that two reflections are equal.
(i) We show the first equality by using the identity $s_1 = s_2 s_3 s_1 s_2 s_3$. If $s_2 s_3 s_1 s_2 s_3$ is admissible on 0, then

$$\mathbf{3} = \overline{\mathbf{1}} = \overline{x_0} = x_{s_2 s_3 s_1 s_2 s_3(0)} = x_{s_1(0)} = x_{|U|} = \mathbf{1}$$

which is a contradiction (Figures 6-a and 7). Thus $s_2 s_3 s_1 s_2 s_3$ is not admissible on 0. Having s_3 not admissible on 0 is impossible since s_3 is admissible



(a) $s_2s_3s_1s_2s_3$ admissible on 0

(b) s_3 not admissible on 0

(c) s_2 not admissible on $s_3(0)$

(d) s_2 not admissible on $|U|$

(e) s_3 not admissible on $s_1s_2s_3(0)$

(f) s_1 not admissible on $s_2s_3(0)$

(f1) $s_2s_3(0) = 0$

(f2) $s_2s_3(0) = |U|$

Figure 6: Cases yielding a contradiction in the proof of Lemma 8. Applying a reflection on a vertex is represented by a plain edge if it is admissible and by a dashed one otherwise.

on everything but $d_1 + d_2$ and $|W| + d_1 + d_2$ (Figure 6-b). Having s_2 not admissible on $s_3(0)$ is also impossible since this implies that $s_3(0) \in I$ and

$$\mathbf{3} = \overline{\mathbf{1}} = \overline{x_0} = x_{s_3(0)} = \mathbf{1}$$

(Figure 6-c). Similar arguments show that supposing s_2 not admissible on $s_3 s_1 s_2 s_3(0)$ or s_3 not admissible on $s_1 s_2 s_3(0)$ yields a contradiction (Figure 6-d-e). Hence, the only remaining possibility is that s_1 is not admissible on $s_2 s_3(0)$ (Figure 6-f). Again there are two cases: either $s_2 s_3(0) = 0$ or $s_2 s_3(0) = |U|$. In the first case, since $s_2 s_3$ fixes 0, it means that $s_2 s_3 = 1$ and then $s_2 = s_3$ which is a contradiction (Figure 6-f1). Otherwise, $s_2 s_3(0) = |U|$ (Figure 6-f2).

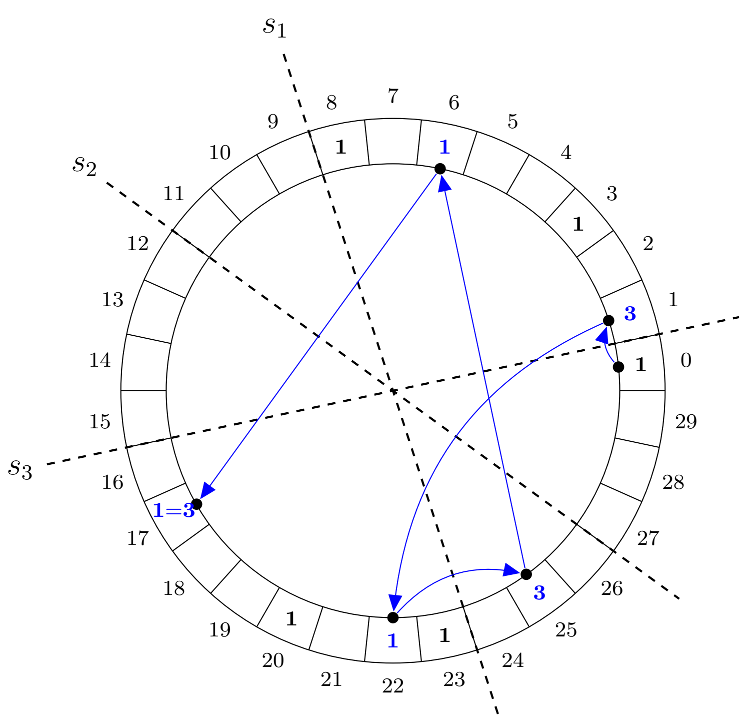


Figure 7: Reflections in action on x where $n = 30$, $d_1 = 3$, $d_2 = 5$, $|U| = 17$, $|X| = 17$ and $|W| = 15$. If $s_2 s_3 s_1 s_2 s_3$ is admissible on 0, then $\mathbf{1} = x_0 = \overline{x_{s_2 s_3 s_1 s_2 s_3(0)}} = \overline{x_{17}} = \overline{\mathbf{1}} = \mathbf{3}$, a contradiction: in fact, any odd-length path between two $\mathbf{1}$'s yields a contradiction.

A similar argument, based on the identity $s_1 = s_3 s_2 s_1 s_3 s_2$ instead of $s_1 = s_2 s_3 s_1 s_2 s_3$, might be used to prove the equality $s_3 s_2(0) = |U|$, concluding part (i). Part (ii) is proved in the same way by considering the identities $s_2 = s_1 s_3 s_2 s_1 s_3$ and $s_2 = s_3 s_1 s_2 s_3 s_1$. $\square$

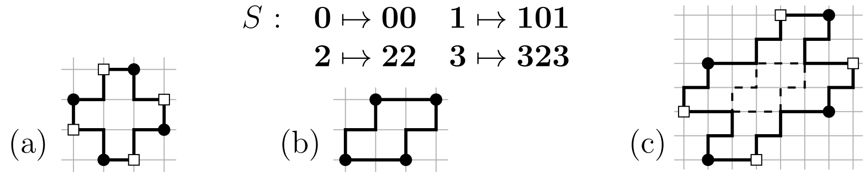
We are now ready to prove the main result.

Proof of Theorem 1. From Lemma 8 (i), we have $s_2 s_3(0) = |U| = s_3 s_2(0)$. Then, $s_3 s_2 = s_2 s_3$ so that s_2 and s_3 must be perpendicular since they are not equal. From Lemma 8 (ii), we have $s_1 s_3(d_1) = d_1 + |X| = s_3 s_1(d_1)$. Then, $s_1 s_3 = s_3 s_1$ so that s_1 and s_3 must be perpendicular since they are not equal. Hence, both s_1 and s_2 are perpendicular to s_3 so that $s_1 = s_2$ which is a contradiction. We conclude that there are no polyomino having three distinct square factorizations of its boundary. $\square$

Notice that in the proof of the main theorem, the contradictions are obtained on the equality of two distinct reflections s_i or on the equality of two distinct corners. This shows that the alternation of square factorizations as stated in Lemma 5 is sufficient but too strong and that a lighter version of it could be used: the proof of Proposition 4 in [7] can be adapted straightforwardly for that purpose.

4. Conclusion

In this paper, we consider tilings by translation of a single polyomino or tile on the square grid $\mathbb{Z}^2$ and we prove that no polyomino admits more than two regular square tilings which was conjectured in 2008. Our approach uses geometrical and topological properties of the boundary word of tiles that are deduced from equations on words. This leads to another conjecture by Provençal and Vuillon [8] (proved in [16]) stating that if $AB\widehat{A}\widehat{B}$ and $XY\widehat{X}\widehat{Y}$ are the BN-factorizations of a prime double square D, then A, B, X and Y are palindromes or equivalently D is invariant under a rotation ρ^2 of 180 degrees (see Figure 2). Note that a polyomino is prime if it is not obtained by composition of smaller square tiles (see Figure 8). Moreover, it would be interesting to extend the results of this paper to piecewise C^2 continuous curves in the way Beauquier and Nivat did for their characterization [17]. The problem of generating efficiently double square tiles is also a problem deserving attention and is addressed in [16]. As a last remark, the method developed in this paper can be adapted to prove that no double square tile admits a hexagonal tiling.



$$S: \quad \mathbf{0} \mapsto \mathbf{00} \quad \mathbf{1} \mapsto \mathbf{101}$$
$$\mathbf{2} \mapsto \mathbf{22} \quad \mathbf{3} \mapsto \mathbf{323}$$

Figure 8: (a) A prime double square D. (b) A square tile S. (c) The tile $S(D)$, which is obtained by replacing each unit square of D by S, is a double square tile. It is not prime.

Acknowledgements

The authors are grateful to the anonymous referees for suggesting useful improvements for the presentation of this paper. Moreover, this is the journal version of a communication presented at the 7th International Conference on Lattice Paths combinatorics and applications [18] which took place in Siena (Italy).

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